astra/1

A Language for Historical Astronomical Models and the Geometrical Reasoning about Them

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Abstract

A historical astronomical model is a geometrical construction with a kinematic variable. Texts about such models do four things at once: they describe a geometry, give it numbers, show it in one position, and reason about it. We present astra/1, a small language that writes each of the four as a text of its own kind. A *model* is the pure geometry: parameter symbols, a kinematic variable, a construction without coordinates, observables; it holds no values. A *parametrisation* gives the parameters values, each with the page it comes from; one model has many. A *state* is a value of the kinematic variable; model, parametrisation and state together give a configuration. A *superposition* places several models in one plane and identifies their inputs. A *derivation* is the operation that follows construction: hypotheses, auxiliary constructions, propositions derived from named premises by named rules, quantities computed or determined, a conclusion. Four questions are asked of a derivation: whether it is grounded, whether every proposition is true, whether every step is an instance of the rule it names with its configuration conditions holding, and whether the conclusion rests on what is cited. Every text names three roles that must not be confused: whose model it is, which text's use of it is analysed, and who wrote the description. We describe Ptolemy's solar model, in its eccentric and its epicyclic form, as Noel Swerdlow uses it in *The Renaissance of Astronomy* (2026). The diagram that shows both forms is a superposition, and their equivalence is the conclusion of a derivation under hypotheses on the radii: equivalence in position when the radii are equal, by Elements I.28 and I.33, and in direction when they are proportional, by VI.6 and I.29. The parallelogram argument does not apply at the apsides, where the parallelogram collapses; the derivation therefore argues by three cases. Ptolemy's reported eccentricity and apogee follow from his rounded arcs of mean motion, not from the day counts of the seasons.

KEYWORDS history of astronomy · Ptolemys solar model · geometrical reasoning · domain-specific language · model equivalence · scholarly reference objects

1 A diagram and what it does not say

Chapter 1 of Noel Swerdlow's *The Renaissance of Astronomy* [1] sets out Ptolemy's solar theory with three diagrams. The first (Fig. 1.2, printed page 20) shows two models of the sun's inequality at once. In the *eccentric* form the sun S moves uniformly on a circle whose centre E is displaced from the earth O. In the *epicyclic* form a small circle, the epicycle, carries the sun; its centre C moves uniformly on a circle about O, while the epicycle turns so that its radius keeps the direction of the line through the apogee. The prose on the facing page says how the two are related: "To construct the epicyclic model, complete parallelogram OESC and let C be the center of an epicycle of radius r = e" on a circle "of radius R′ = R" (p. 19). The models are Ptolemy's; Swerdlow says so in the same paragraph ("Ptolemy sets out eccentric and epicyclic models for this inequality"). What we analyse is Swerdlow's use of them.

A reader who wants to understand the diagram has to take apart four things that the page presents together.

A geometry. Each form is a construction: points, circles, an angle that varies. Nothing in it depends on how large the eccentricity is.

Numbers. The values R = 60, e = 2;29,30 ≈ 2;30 and λ_A = 65;30° stand two pages later (p. 22), and they are Ptolemy's. The same paragraph gives other values for the same geometry, al-Battānī's. The printed diagrams use a third set: an eccentricity many times too large, so that the drawing can be read.

A position. The diagram shows the construction for one value of the mean anomaly, about 41°. The same geometry with the same numbers appears again in Fig. 1.4 in another position and another orientation.

An argument. The diagram shows *two* models in one plane, and the claim about them is that they are equivalent. The parallelogram is not a part of either model. It is the place of an argument, and the argument has hypotheses, r = e and R′ = R, that the diagram cannot show.

An earlier version of this paper (1.1) did not keep these apart. It wrote geometry, Ptolemy's values and the depicted position into one text; it described Fig. 1.2 as a "correspondence" from one model to the other with a "parameter map"; it checked the steps of the argument only numerically; and it named Swerdlow, whose exposition it followed, as the author of the models. This version corrects all four. The design now follows five requirements.

1. **A model has no values.** The geometry is one text, checked over its whole parameter space. Values are another text, with an owner and sources. A position is an argument.
2. **Three roles are named and never mixed**: whose model it is, the text whose use of it is analysed, and who wrote the description.
3. **A diagram of two models is a superposition**, and what holds across the two is derived under stated hypotheses, not drawn in.
4. **Geometrical reasoning is an operation of its own**, after the construction: a derivation whose steps name their premises and their rules, and each step is checked for truth and for whether the rule licenses it.
5. **Every part is citable**, down to a step of a derivation.

Section 2 introduces the vocabulary with the eccentric form as the running example. Section 3 gives the language, Section 4 its meaning, Section 5 what can be said of each kind of text. Section 6 treats superposition and equivalence, Section 7 derivation. Section 8 describes what follows from the texts, Section 9 the case study and what it found, Section 10 the relation of model, representation and description. Section 11 compares astra/1 with the work it learned from; Section 12 states what is not done.

2 Model, parametrisation, state

Three questions can be asked of a historical astronomical model, and their answers are of different kinds.

What is its geometry? The eccentric form is: a circle whose centre E lies at some distance from the observer O in some direction; a point S on the circle at an angle $\bar{\kappa}$ from the apogee; the direction from O to S as what is observed. Relations that follow from the construction alone — the line through apogee and perigee passes through E and O; the mean longitude is the longitude of the apogee plus the mean anomaly — are true for every eccentricity, every radius, every direction of the apogee, in every position. We call this the **model**. It has parameter *symbols* and a **kinematic variable**, $\bar{\kappa}$, and no values.

What are its values? Ptolemy found R = 60, e = 2;30 and λ_A = 65;30°. With them the model has numerical consequences: the largest equation of centre is 2;23°. Al-Battānī found e = 2;4,45 and λ_A = 82;17°; the largest equation is then 1;59°. A set of values for a model's parameters, each with the place where it stands and with the name of whose values they are, is a **parametrisation**. One model has many.

Where is everything? That needs, besides values, a value of the kinematic variable: a **state**. A model under a parametrisation in a state is a **configuration**: every point has a place. A diagram shows one configuration; an animation runs through the states.

The three fix different things, stand in different places and support different statements (Table 1).

Table 1 Model, parametrisation and state.

	what it fixes	where it stands	what can be said of it
model	parameter symbols with units and admissible ranges; the kinematic variable; the construction; the observables	a `model` text	what holds for every admissible parameter value and every state
parametrisation	the values of the parameters, each with source or stated gap; whose values they are	a `parametrisation` text	what follows for these values
state	the value of the kinematic variable, or a time	an argument, `at kappa_bar = 41°`	where everything is: a configuration

Two more kinds of text do something with models. A **superposition** places several models in one plane (Section 6). A **derivation** reasons in a model or a superposition (Section 7). A fifth kind, the **figure**, says what is drawn: which context, which values, which state, which labels. Figure 1 shows how the kinds refer to one another.

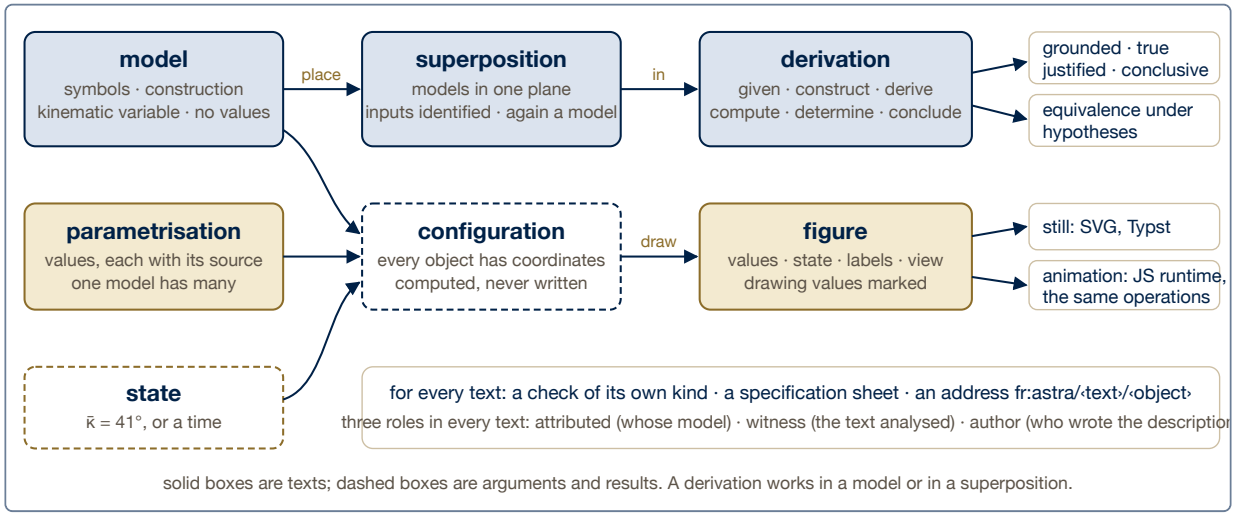


Figure 1 The kinds of text of astra/1 and what they refer to.

Every text carries three names. `attributed` says whose model, or whose values, it is. `witness` gives the address of the text whose use of the model is analysed. `author` says who wrote the astra text. For the texts of this paper these are Ptolemy, Swerdlow's pages, and the author of this paper. An exposition is not an authorship: the language refuses an `author` that is a source address, and a model without `attributed` is not documented.

3 The language

3.1 Texts

An astra/1 text is a sequence of statements, one per line; the first names the kind of text. A `#` begins a comment. Numbers are written as the sources write them: decimal (`12.5`) or sexagesimal, with a semicolon after the integer part and commas between the fractional places, so that `2;30` is 2 + 30/60 and `365;14,48` is 365 + 14/60 + 48/3600. A degree sign makes an angle (`65;30°`). The written form of every number is kept. Identifiers are ASCII letters, digits, underscores and the prime (`S'`, `R'`); the printed symbol is declared separately (`display kappa_bar "κ"`). Addresses of sources are written `fr:…` (Section 8.3).

Listing 1 is the complete model text of the eccentric form, comments removed. It contains no number that is a value.

Listing 1 The model `solar-eccentric`: Ptolemy's solar model in its eccentric form. Parameters are symbols with units and ranges; `variable` declares the kinematic variable, `law` its link to time; `proposition` states what is then checked over the parameter space.

```
model solar-eccentric "Ptolemy's solar model, eccentric form"
attributed "Ptolemy"
witness fr:kepler-sec/Swerdlow2026/0049/p04 "the construction, p. 19"
witness fr:kepler-sec/Swerdlow2026/0050/p03 "longitudes, anomalies and the
      equation, p. 20"
witness fr:kepler-sec/Swerdlow2026/0051/fig02 "Fig. 1.4"
author "Gerd Graßhoff"
tradition ptolemaic
frame geocentric
param R  : length "radius of the eccentric"  in (1, 1000)
param e  : length "eccentricity, the distance OE"  in (0, R / 2)
param lamA  : angle  "longitude of the apogee"
param T  : time  "tropical year"  in (300, 400)
param kappa_0 : angle  "mean anomaly at t = 0"
display lamA "λ_A"
display kappa_0 "κ̄0"
variable kappa_bar : angle "mean anomaly about E, counted from the apogee"
law kappa_bar = uniform(period: T, epoch: kappa_0) "the sun moves uniformly
      about E"
display kappa_bar "κ"
O  = observer  "earth"
E  = O + polar(e, lamA)  "centre of the eccentric"
ecc = circle(E, R)  "the eccentric"
A  = E + polar(R, lamA)  "apogee"
B  = E - polar(R, lamA)  "perigee"
S  = E + polar(R, lamA + kappa_bar)  "sun"
body sun at S
observe lam_bar = direction(E, S)  "mean longitude"
observe lam  = direction(O, S)  "true longitude"
observe kappa  = lam - lamA  "true anomaly about O"
observe c  = lam_bar - lam  "equation of centre, mean minus true longitude"
display lam_bar "λ̄S"
display lam "λ_S"
display kappa "κ"
proposition p1: equal(lam_bar, lamA + kappa_bar) "the mean longitude is the
      longitude of the apogee plus the mean anomaly"
proposition p2: equal(kappa, kappa_bar - c)  "the true anomaly is the mean
      anomaly less the equation"
proposition p5: equal(lam, lam_bar - c)  "the true longitude is the mean
      longitude less the equation"
proposition p3: collinear(A, E, B)  "the apsidal line passes through E"
proposition p4: collinear(A, O, B)  "and through O"
```

The values are other texts (Listing 2). The second is partial: the paragraph that reports al-Battānī's values gives no year and no epoch, and the text does not invent them.

Listing 2 Two parametrisations of the one model. `0052/p05` is block 5 of scan page 52, printed page 22; the reference register counts the running head as the first block. A value without a source needs a `gap` that says why.

```
parametrisation solar-eccentric.ptolemy of solar-eccentric "Ptolemy's values"
attributed "Ptolemy"
witness fr:kepler-sec/Swerdlow2026/0052/p05
author "Gerd Graßhoff"
R  = 60  "the norm of the radius"  source fr:kepler-sec/Swerdlow2026/0052/p05
e  = 2;30  "2;29,30 rounded to 2;30"  source fr:kepler-sec/Swerdlow2026/0052/p05
lamA  = 65;30°  source fr:kepler-sec/Swerdlow2026/0052/p05
T  = 365;14,48  source fr:kepler-sec/Swerdlow2026/0047/p03
kappa_0 = 0°  gap "the pages give no epoch; t is counted from a passage through
      the apogee"

parametrisation solar-eccentric.battani of solar-eccentric "al-Battānī's values"
attributed "al-Battānī"
witness fr:kepler-sec/Swerdlow2026/0052/p06
author "Gerd Graßhoff"
R  = 60  gap "the norm R = 60 is not repeated in the paragraph; e is given in
      its units"
e  = 2;4,45  source fr:kepler-sec/Swerdlow2026/0052/p06
lamA = 82;17°  source fr:kepler-sec/Swerdlow2026/0052/p06
```

3.2 Statements

Table 2 lists the statements by kind of text. Each has a fixed form. A statement in a text of the wrong kind is refused with a message that says where it belongs (Listing 3).

Table 2 The statements of astra/1 by kind of text; the full forms are in the specification.

kind	statements
all	`attributed`, `witness`, `author`, `display`
model	`param … : <unit> [in (lo, hi)]`, `variable`, `law`, construction `<name> = <expr>`, `observe`, `body`, `proposition`, `tradition`, `frame`
parametrisation	value `<param> = <expr>` with `source fr:…` or `gap "why"`, `use`
superposition	`place <model> as <Prefix>`, `identify <P.x> = <Q.y>`
figure	`with <parametrisation>`, `drawn with`, `at <variable> = <angle>`, `up`, `fit`, `speed`, `label`, `hide`
derivation	`given`, `construct`, `derive … from … by …`, `compute`, `determine`, `conclude`

Listing 3 What the parser and the superposition refuse, with their messages.

```
param R = 60 length "radius"
  → a model holds no values: declare `param R : <unit>` here and give the value
       in a parametrisation
R = 60 source fr:kepler-sec/Swerdlow2026/0052/p05
  → a model holds no values and no sources; values and their sources stand in a
       parametrisation
at kappa_bar = 41°
  → `at` belongs in a figure text, not in a model
author fr:kepler-sec/Swerdlow2026
  → `author` names who wrote this text; the text analysed is the `witness`
identify Ecc.S = Epi.S'
  → only parameters of one unit, kinematic variables and the observer can be
```

```
        identified; that two constructed points coincide is a proposition to
        derive
```

Expressions use the four arithmetic operators, parentheses, names and functions (Table 3). The predicates of propositions are `parallel`, `perpendicular`, `coincident`, `collinear`, `on` and `equal`.

Table 3 The functions of astra/1.

function	gives	meaning
`observer`	point	the origin of the model frame
`polar(r, θ)`	displacement	length r in direction θ; `P + polar(r, θ)` is a point
`circle(P, r)`, `line(P, Q)`, `ray(P, θ)`	circle, line	lines are directed
`intersect(k, l, ±)`	point	circle with line or circle; the sign chooses the branch
`foot(P, l)`, `midpoint(P, Q)`, `rotate(P, about: A, by: θ)`, `reflect(P, l)`	point	
`ellipse(F₁, F₂, a)`, `on_ellipse(k, E)`, `tangent(P, k, ±)`, `seen_from(Q, k, θ)`	ellipse, point	Kepler's orbit; the equant
`direction(P, Q)`, `distance(P, Q)`	angle, length	undefined when P and Q coincide
`angle(P, V, Q)`	angle	the unsigned angle at V, between 0° and 180°: the angle of Euclid's propositions
`true_anomaly(M, e)`	angle	from Kepler's equation
`sin`, `cos`, `tan`, `asin`, `atan2`, `sqrt`, `abs`, `norm360`, `norm180`	number, angle	

An intersection of a circle and a line has two points, and a model must say which one it means: for a circle and a directed line, `+` is the point farther along the line's direction; for two circles, the point to the left of the line from the first centre to the second. A missing sign is an error, not a guess.

3.3 Types and the abstract syntax

Every name and every expression has a type: a number, an angle, a length, a time, a rate, a point, a displacement, a line, a circle or an ellipse. An angle cannot be added to a length; a circle needs a point and a length; a product of two lengths is allowed on the way to a length or in a comparison of squares. Names are defined before they are used, so a construction is a directed graph without cycles by the way it is written, as in GCLC [2]. Nothing is placed by coordinates except the observer, the origin.

What the tools work on is the *abstract syntax*: the tree of statements and expressions that the parser builds, independent of spacing and comments. A superposition merges the trees of its members; a derivation extends the tree of its context; a rule of derivation is a pair of trees with variables (Section 7).

4 What a model means

A context — a model, a superposition, or a derivation with its auxiliary constructions — compiles to a list of *operations* in the order in which they must be computed. Each operation has a name, the names of the operations it reads, and a type. The operation set has forty-two operations (Table 4) and is the only place where the geometry is defined. A value is a number or a short list of numbers: a point is its two coordinates, a circle its centre and radius, a line a point and a unit direction. Coordinates appear here, inside the meaning of a model, never in its text.

Table 4 The forty-two operations (`spec/ops.yaml`), grouped.

operations	what they compute
`param`, `var`	the inputs: a parameter, read from the parametrisation; a kinematic variable, read from the state
`const`, `id`	a literal; a named value
`add`, `sub`, `mul`, `div`, `neg`, `abs`, `norm360`, `norm180`	arithmetic on numbers and displacements
`observer`, `polar`, `circle`, `line`, `ray`	elementary placements
`x_cl`, `x_cc`, `x_el`, `tangent`, `seen_from`	intersections and contacts; no solution gives "not a number"
`foot`, `midpoint`, `rotate`, `reflect`, `ellipse`, `on_ellipse`	further constructions
`direction`, `distance`	what an observer measures; undefined for coincident points
`kepler_E`, `true_anom`	Kepler's equation, solved by Newton's method
`sin`, `cos`, `tan`, `atan2`, `asin`, `sqrt`	functions for formulae
`res_parallel`, `res_perp`, `res_collinear`, `res_on`	residuals of predicates

There is one evaluation. Given the operation list of a context, parameter values p and a state s, it returns the value of every object:

configuration(operations, p, s).

The model *is* this function with p and s left open. A parametrisation fixes p. A state fixes s, either by naming the kinematic variable ($\bar{\kappa} = 41°$) or by a time, which the variable's law turns into a value: for the uniform law, $\bar{\kappa} = \bar{\kappa}_0 + 360° \cdot t / T$, with the period T and the epoch $\bar{\kappa}_0$ as ordinary parameters. Nothing has a default. A parameter without a value is an error as soon as anything other than a range or a law needs it; a state that fixes no variable is an error. So a partial parametrisation, like al-Battānī's, can still be used for what it determines — the largest equation needs only R and e — and says what it cannot give.

Because a model has no values, statements about it are tested where it lives. The parameter space is sampled at 48 points, each parameter in its declared range; a range that depends on another parameter, like that of the eccentricity, `in (0, R / 2)`, is evaluated inside each sample. The states are the kinematic variable in 72 steps of 5°. A proposition is turned into a *residual*, a number that is zero when it holds: the distance of two points (`coincident`), the sine of the angle between two lines (`parallel`), the cosine (`perpendicular`), the sine of the bend at the middle of three points (`collinear`), the absolute difference of two scalars, angles reduced to the range from −180° to 180° (`equal`). A proposition holds when its residual stays within 10^{-9} in all 3,456 configurations, lengths taken relative to the largest length parameter. This is evidence, not proof.

5 What can be said of a text

Each kind of text has its own check, in levels that presuppose one another (Table 5). The report gives the level reached and, for the first level not reached, every reason.

Table 5 The checks by kind of text.

kind	levels
model	well-formed · constructible · astronomical · kinematic · consistent · documented
parametrisation	well-formed · complete · admissible · attested
superposition	well-formed · constructible · superposed · consistent
figure	well-formed · drawable · labelled
derivation	well-formed · grounded · true · justified · conclusive

A model is *constructible* when every object has a solution in every sampled configuration; *astronomical* when it has one observer, a body, an observable direction from the observer to the body and a declared frame; *kinematic* when a kinematic variable reaches a body through the construction; *consistent* when its stated propositions hold in every sampled configuration; *documented* when it names whose model it is, a witness and an author, and gives every symbol and object a meaning. A parametrisation is *complete* when every parameter has a value, *admissible* when every value lies in its range, *attested* when every value has a source or a stated gap. Listing 4 shows two reports.

Listing 4 The reports for the eccentric model and for al-Battānī's parametrisation. The second is incomplete, and says which values the page does not give.

```
$ famulus-astra check solar-eccentric
solar-eccentric: model, level documented (6/6)
  ✓ well-formed
  ✓ constructible
  ✓ astronomical
  ✓ kinematic
  ✓ consistent
  ✓ documented
  ✓ p1: equal(lam_bar, lamA + kappa_bar)  max residual 5.7e-14 over 3456
        configurations
  ✓ p2: equal(kappa, kappa_bar - c)  max residual 1.1e-13 over 3456
        configurations
  ✓ p5: equal(lam, lam_bar - c)  max residual 5.7e-14 over 3456 configurations
  ✓ p3: collinear(A, E, B)  max residual 1.5e-16 over 3456 configurations
  ✓ p4: collinear(A, O, B)  max residual 1.3e-16 over 3456 configurations

$ famulus-astra check solar-eccentric.battani
solar-eccentric.battani: parametrisation, level well-formed (1/4)
  ✓ well-formed
  ✗ complete
      T (tropical year) has no value
      kappa_0 (mean anomaly at t = 0) has no value
  · admissible (not reached)
  · attested (not reached)
  gap: R = 60: the norm R = 60 is not repeated in the paragraph; e is given in
        its units
```

A proposition that is true for Ptolemy's values and not in general, such as that the eccentricity is a twenty-fourth of the radius, is not a property of the model. Stated in the model it fails the level *consistent*, with the sampled values that break it. It belongs to a parametrisation, as a computed consequence (Section 7).

6 Superposition and equivalence

Fig. 1.2 shows the two forms of Ptolemy's solar model in one plane. That is a superposition (Listing 5).

Listing 5 The superposition `solar-eccentric-epicyclic`. It is the scene of Figs. 1.2 and 1.3.

```
superposition solar-eccentric-epicyclic "Eccentric and epicyclic form of
      Ptolemy's solar model, superposed"
witness fr:kepler-sec/Swerdlow2026/0050/fig01 "Fig. 1.2"
witness fr:kepler-sec/Swerdlow2026/0051/fig01 "Fig. 1.3"
author "Gerd Graßhoff"
place solar-eccentric as Ecc
place solar-epicyclic as Epi
identify Ecc.O = Epi.O  "one earth"
identify Ecc.lamA = Epi.lamA  "one apsidal direction"
identify Ecc.T = Epi.T  "one year"
identify Ecc.kappa_0 = Epi.kappa_0  "one epoch"
identify Ecc.kappa_bar = Epi.kappa_bar "one mean anomaly"
display Epi.lam "λ'_S"
display Ecc.lam "λ_S"
```

Every name of a member gets the member's prefix (`Ecc.S`, `Epi.S'`). The names the text identifies become one: the observer, the longitude of the apogee, the year, the epoch, the mean anomaly. Identification says that the two models are about the same earth, the same apsidal direction and the same mean motion. It adds no geometry. Only inputs — parameters of one unit, kinematic variables — and the observer can be identified. A text that identifies the two suns is refused (Listing 3): that two constructed points coincide is a proposition, and propositions are derived.

A superposition is again a model: pure geometry over the parameters of both members, checked like any model. In it the two suns are two points. For some parameter values they coincide, for others they lie on one ray from the earth, for most they do neither. Nothing in the superposition says which. Figure 2 shows the two members separately, Figure 3 the superposition under the values of Fig. 1.2 and of Fig. 1.3.

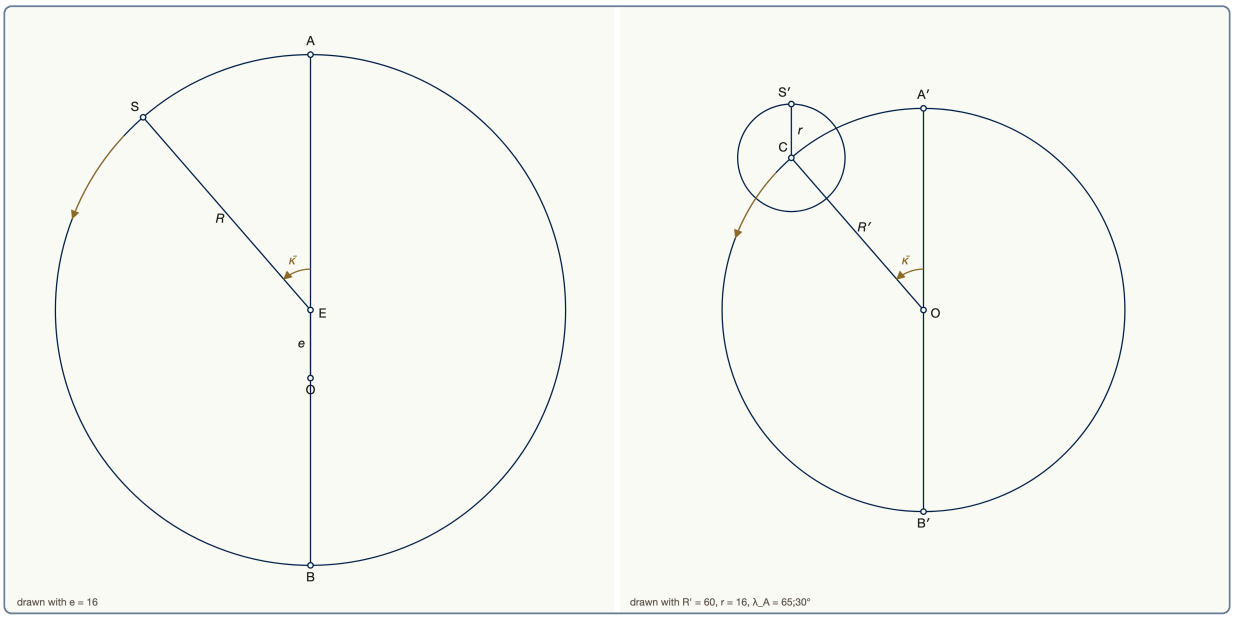


Figure 2 The two members drawn separately: the eccentric form (left) and the epicyclic form (right). The pages give the epicyclic form no values of its own; it is drawn with drawing values, and the corner says so.

Equivalence is a relation between two models. It holds relative to hypotheses and relative to what is compared. The pages contain two equivalences of the same two models:

Table 6 The two equivalences of the eccentric and the epicyclic form.

	hypotheses	the models are equivalent in
Fig. 1.2	R′ = R and r = e	*position*: the two suns are one point in every state
Fig. 1.3	R′/r = R/e	*direction*: the sun is seen from the earth in the same direction; the distances differ

Each is the conclusion of a derivation in the superposition. Equivalence is distinct from **identity**. Two texts are the same model when they are the same geometry up to renaming, which a structural comparison tests (Section 8.4). The eccentric and the epicyclic form are different models. Under hypotheses they agree in what is compared, and they stay different in the rest: in the one the sun moves uniformly on a circle about E, in the other its motion is compounded of two. Swerdlow notes that only Kepler drew a consequence from this difference (p. 20).

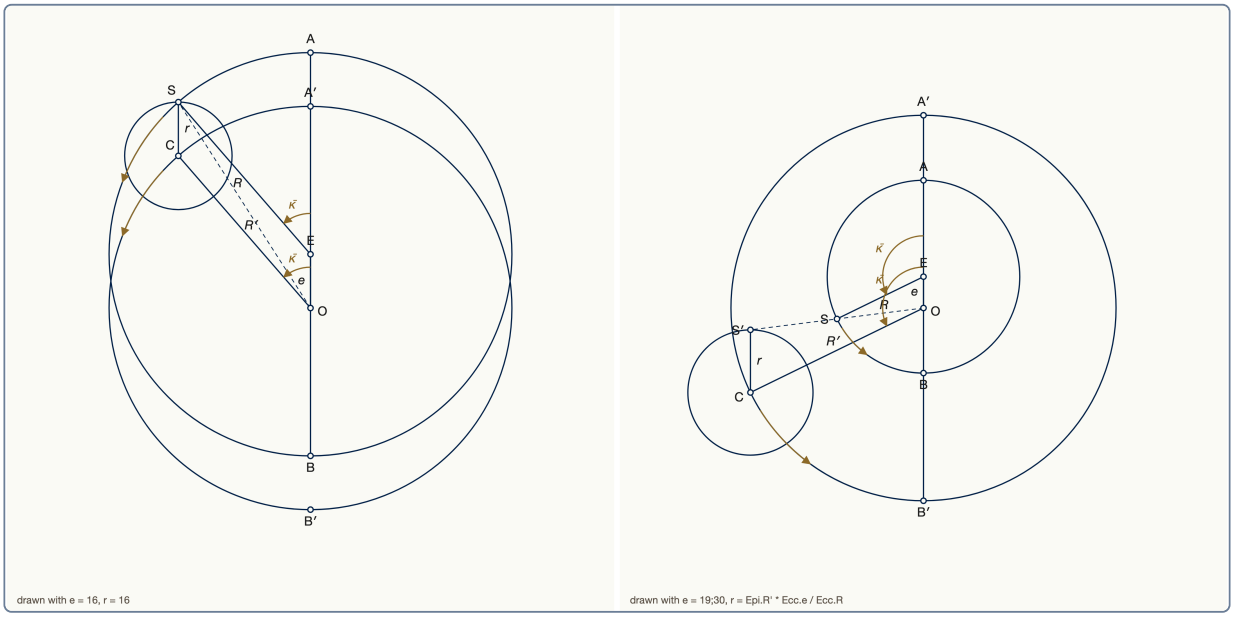


Figure 3 The superposition under two parametrisations. Left, the values of Fig. 1.2 (R′ = R, r = e): the two suns are one point. Right, the values of Fig. 1.3 (R′/r = R/e): two suns on one ray from the earth. Both are drawn with an exaggerated eccentricity, as the printed figures are.

7 Derivation

After the construction of a model comes the operation of derivation: geometrical reasoning about what has been constructed. A derivation works in a context, a model or a superposition, and consists of operations (Table 7).

Table 7 The operations of a derivation.

operation	what it does
`given h: equal(<param>, <expr>)`	a hypothesis; it restricts the parameter space
`construct F = <expr>`	an auxiliary construction, added to the context

operation	what it does
`derive d: <proposition> from <premises> by <rule>`	a proposition derived from named premises by a named rule
`show A: <proposition> … qed from <steps>`	a sub-argument: the claim first, then the steps that establish it; its inside cannot be cited from outside
`cases` / `case n: when <condition>` / `qed by cases`	a claim argued separately under conditions that together must cover every configuration
`suppose n: <condition>`	restricts the steps that follow to the configurations in which the condition holds
`conclude <proposition> from <premises>`	the proposition the derivation is for; `equivalent(P, Q, position)` or `equivalent(P, Q, direction)` for an equivalence of two members
`compute x = <expr>`	a quantity under a parametrisation, compared with a reported value
`determine p, q from …`	parameters determined from observed intervals through the model

A premise is a given or an earlier step, by name; the construction of an object (`def Ecc.S`); or a proposition stated in the model (`prop p1`). A rule is a proposition of Euclid (`Elements I.33`), a lemma, a definition of trigonometry, `common notion 1`, `construction` or `algebra`. The propositions of a step are those of a model, with two more: `less(a, b)`, and `approximately(a, b, small: <ratio>, order: n)` for a step that holds to a stated order in a small ratio (rule `expansion`). The notions are set out together in [3]; sub-arguments, cases and suppositions were added to the language for the reconstructions in [4] and [5].

7.1 The derivation that Fig. 1.2 exhibits

The page names the parallelogram OESC. It does not give the steps. Listing 6 gives them, with the rule that licenses each. The claim that the two suns are one point is argued in three cases: off the apsidal line, at the apogee, at the perigee.

Listing 6 The derivation `solar-equivalence-equal-radii`, comments removed. Ecc and Epi are the two members of the superposition.

```
derivation solar-equivalence-equal-radii in solar-eccentric-epicyclic "The two
      forms place the sun at the same point when R' = R and r = e"
witness fr:kepler-sec/Swerdlow2026/0049/p04 "the parallelogram OESC, p. 19"
witness fr:kepler-sec/Swerdlow2026/0050/fig01 "Fig. 1.2"
author "Gerd Graßhoff"
given h1: equal(Epi.R', Ecc.R) "the deferent has the radius of the eccentric"
given h2: equal(Epi.r, Ecc.e)  "the epicycle has the eccentricity as its radius"
show same: coincident(Ecc.S, Epi.S') "the sun of the eccentric form and the sun
      of the epicyclic form are one point"
  derive d3: equal(distance(Ecc.E, Ecc.S), Ecc.R)  from def Ecc.S by
       construction  "S lies on the eccentric"
  derive d4: equal(distance(O, Epi.C), Epi.R')  from def Epi.C by construction
       "C lies on the deferent"
  derive d8: equal(distance(O, Ecc.E), Ecc.e)  from def Ecc.E by construction
       "OE is the eccentricity"
  derive d9: equal(distance(Epi.C, Epi.S'), Epi.r)  from def Epi.S' by
       construction "S' lies on the epicycle"
```

```
cases "by the position of the sun: off the apsidal line, at the apogee, at the
      perigee"
case generic: when noncollinear(O, Ecc.E, Ecc.S) "off the apsidal line, OESC
      is a parallelogram"
  show sides: parallel(line(Ecc.E, Ecc.S), line(O, Epi.C)) "ES and OC are
        parallel"
    derive d1: equal(angle(Ecc.A, Ecc.E, Ecc.S), angle(Epi.A', O, Epi.C)) from
          def Ecc.A, def Ecc.S, def Epi.A', def Epi.C by construction "both
          angles are the mean anomaly κ̄, counted from the apsidal line"
    derive d2: parallel(line(Ecc.E, Ecc.S), line(O, Epi.C)) from d1 by
          Elements I.28 "the apsidal line falls on ES and OC and makes equal
          angles with them on the same side"
  qed from d2
  derive d5: equal(distance(Ecc.E, Ecc.S), distance(O, Epi.C)) from d3, d4, h1
        by common notion 1 "ES = R = R' = OC"
  show joins: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) "OE and CS are
        parallel: OESC is a parallelogram"
    derive d6: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) from sides, d5 by
          Elements I.33 "the lines joining the ends of the equal and parallel
          ES and OC"
  qed from d6
  derive d7: equal(distance(O, Ecc.E), distance(Epi.C, Ecc.S)) from sides, d5
        by Elements I.33 "and they are equal"
  derive d10: equal(distance(Epi.C, Ecc.S), distance(Epi.C, Epi.S')) from d7,
        d8, h2, d9 by common notion 1 "CS = OE = e = r = CS'"
  derive d11: parallel(line(Epi.C, Epi.S'), line(O, Ecc.E)) from def Epi.S',
        def Ecc.E by construction "the radius of the epicycle keeps the
        direction of the apsidal line"
  derive d12: coincident(Ecc.S, Epi.S') from joins, d11, d10 by lemma same ray
        "S and S' lie from C along the parallel to OE, the same way, at the
        same distance"
qed from d12
case apogee: when collinear(O, Ecc.E, Ecc.S), between(Ecc.E, O, Ecc.S) "at the
      apogee the four points lie on the apsidal line, E between O and S"
  derive a1: equal(distance(O, Ecc.S), distance(O, Ecc.E) + distance(Ecc.E,
        Ecc.S)) by lemma segment addition "OS = OE + ES"
  derive a2: equal(distance(O, Epi.S'), distance(O, Epi.C) + distance(Epi.C,
        Epi.S')) by lemma segment addition "OS' = OC + CS'"
  derive a3: equal(distance(O, Ecc.S), distance(O, Epi.S')) from a1, a2, d3,
        d4, d8, d9, h1, h2 by algebra "e + R = R' + r"
  derive a4: parallel(line(O, Ecc.S), line(O, Ecc.E)) from apogee by lemma one
        line "S lies on the apsidal line"
  derive a5: collinear(O, Ecc.E, Epi.S') from def Epi.C, def Epi.S', def Ecc.E
        by construction "and so does S'"
  derive a6: parallel(line(O, Epi.S'), line(O, Ecc.E)) from a5 by lemma one
        line
  derive a7: coincident(Ecc.S, Epi.S') from a4, a6, a3 by lemma same ray "S
        and S' lie from O along the apsidal line, the same way, at the same
        distance"
qed from a7
case perigee: when collinear(O, Ecc.E, Ecc.S), between(O, Ecc.E, Ecc.S) "at
      the perigee the four points lie on the apsidal line, O between E and S"
  derive p1: equal(distance(Ecc.E, Ecc.S), distance(Ecc.E, O) + distance(O,
        Ecc.S)) by lemma segment addition "ES = EO + OS"
  derive p2: equal(distance(Epi.C, O), distance(Epi.C, Epi.S') +
        distance(Epi.S', O)) by lemma segment addition "CO = CS' + S'O"
```

```
    derive p3: equal(distance(O, Ecc.S), distance(O, Epi.S')) from p1, p2, d3,
          d4, d8, d9, h1, h2 by algebra "R − e = R' − r"
    derive p4: parallel(line(O, Ecc.S), line(O, Ecc.E)) from perigee by lemma
          one line "S lies on the apsidal line"
    derive p5: collinear(O, Ecc.E, Epi.S') from def Epi.C, def Epi.S', def Ecc.E
          by construction "and so does S'"
    derive p6: parallel(line(O, Epi.S'), line(O, Ecc.E)) from p5 by lemma one
          line
    derive p7: coincident(Ecc.S, Epi.S') from p4, p6, p3 by lemma same ray "S
          and S' lie from O along the apsidal line, the same way, at the same
          distance"
  qed from p7
qed by cases
conclude equivalent(Ecc, Epi, position) from same "in every state the two forms
      have the sun at one point"
```

In words: the angles AES and A′OC are both the mean anomaly, so ES and OC are parallel (Elements I.28: “if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side … the straight lines will be parallel”). They are equal, since R = R′. So the lines that join their ends, OE and CS, are equal and parallel (I.33: “the straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel”) [6]. The sun S′ of the epicyclic form lies from C in the direction of the apsidal line at the distance r, and r = e. So S and S′ lie from C along the same parallel, the same way, at the same distance: they are one point. That is the first case. At the apogee and at the perigee the four points lie on the apsidal line and there is no parallelogram; there OS = e + R and OS′ = R′ + r (at the perigee R − e and R′ − r), which are equal under the two hypotheses, and both suns lie on the apsidal line on the same side of O. Figure 4 shows three steps of the first case in the figure.

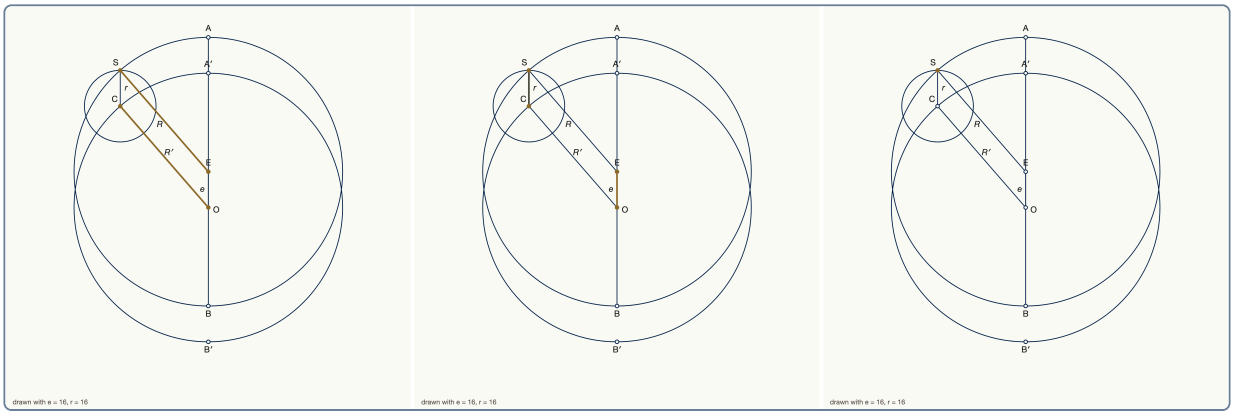


Figure 4 Three steps in the scene of the derivation, each with the points and segments it speaks of in gold: d2, ES is parallel to OC; d6, OE is parallel to CS; d12, S and S′ are one point.

7.2 Four questions

Four questions are asked of a derivation, each presupposing the one before.

Is it grounded? Every premise resolves to a given, an earlier step, a construction or a stated proposition.

Is it true? Every proposition holds in all sampled configurations that satisfy the givens. A given of the form `equal(<parameter>, <expression>)` fixes that parameter by the others, so the samples lie on the part of the parameter space where the hypotheses hold.

Is it justified? Every step must be licensed by the rule it names. There are four grades.

- *An instance of a rule.* A rule is a schema (Table 8): variables for points, hypotheses, conclusions. A step is an instance when the variables can be bound to points of the context so that a conclusion of the rule is the step's proposition and each hypothesis of the rule is one of the cited premises. The match respects the symmetries of the predicates: `parallel` and `equal` are symmetric, a line and a distance do not depend on the order of their points, an angle is unsigned. For step d6 the binding is A = O, B = C, C = E, D = S, and the two hypotheses of I.33 are met by d2 and d5.
- *Common Notion 1.* The two sides of an equality are joined by the cited equalities.
- *Read off the construction.* The proposition speaks only of the cited objects and what they are built from, and it holds over the whole parameter space **without** the givens. That ES = R is read off the construction of S. A proposition that needs a hypothesis cannot pass as read off.
- *Calculation.* An equality from cited equalities. It is confirmed numerically; the calculation is not replayed.

A sub-argument is justified when the steps it closes with establish its claim, and an argument by cases when every case reaches the claim and the conditions of the cases cover every sampled configuration. The steps of a case are checked only in the configurations that satisfy its condition. A step that is true and none of these is reported as *true, not justified*: a gap in the reasoning.

Is it conclusive? For every given, the parameter space is sampled once more with that given lifted. A proposition that then fails *needs* the given. If the given is not cited on the way to that proposition, it is a **tacit premise**, and the derivation is not conclusive. Givens that the conclusion does not need, and steps that are not on the way to it, are noted.

Listing 7 is the report for the derivation of Fig. 1.2, and Listing 8 what the checks say for six altered versions of it.

Listing 7 The report for `solar-equivalence-equal-radii`. Of the twenty-six steps, sixteen are instances of rules or of Common Notion 1, eight are read off the constructions, two are calculations.

```
$ famulus-astra derive solar-equivalence-equal-radii
solar-equivalence-equal-radii: derivation in solar-eccentric-epicyclic, level
      conclusive (5/5)
  ✓ well-formed
  ✓ grounded
  ✓ true
  ✓ justified
  ✓ conclusive
  given  h1: equal(Epi.R', Ecc.R)
  given  h2: equal(Epi.r, Ecc.e)
  ✓ same: coincident(Ecc.S, Epi.S') by cases — justified by cases
    ✓ d3: equal(distance(Ecc.E, Ecc.S), Ecc.R) from def Ecc.S by construction —
          read off the construction
    ✓ d4: equal(distance(O, Epi.C), Epi.R') from def Epi.C by construction —
          read off the construction
    ✓ d8: equal(distance(O, Ecc.E), Ecc.e) from def Ecc.E by construction — read
          off the construction
    ✓ d9: equal(distance(Epi.C, Epi.S'), Epi.r) from def Epi.S' by construction
          — read off the construction
    ✓ generic: coincident(Ecc.S, Epi.S') from d12 — justified (it is d12)
      ✓ sides: parallel(line(Ecc.E, Ecc.S), line(O, Epi.C)) from d2 — justified
            (it is d2)
        ✓ d1: equal(angle(Ecc.A, Ecc.E, Ecc.S), angle(Epi.A', O, Epi.C)) from
              def Ecc.A, def Ecc.S, def Epi.A', def Epi.C by construction — read
              off the construction
        ✓ d2: parallel(line(Ecc.E, Ecc.S), line(O, Epi.C)) from d1 by Elements
              I.28 — justified
      ✓ d5: equal(distance(Ecc.E, Ecc.S), distance(O, Epi.C)) from d3, d4, h1 by
```

```
            common notion 1 — justified
      ✓ joins: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) from d6 — justified
            (it is d6)
        ✓ d6: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) from sides, d5 by
              Elements I.33 — justified
      ✓ d7: equal(distance(O, Ecc.E), distance(Epi.C, Ecc.S)) from sides, d5 by
            Elements I.33 — justified
      ✓ d10: equal(distance(Epi.C, Ecc.S), distance(Epi.C, Epi.S')) from d7, d8,
            h2, d9 by common notion 1 — justified
      ✓ d11: parallel(line(Epi.C, Epi.S'), line(O, Ecc.E)) from def Epi.S', def
            Ecc.E by construction — read off the construction
      ✓ d12: coincident(Ecc.S, Epi.S') from joins, d11, d10 by lemma same ray —
            justified
    ✓ apogee: coincident(Ecc.S, Epi.S') from a7 — justified (it is a7)
      ✓ a1: equal(distance(O, Ecc.S), distance(O, Ecc.E) + distance(Ecc.E,
            Ecc.S)) by lemma segment addition — justified
      ✓ a2: equal(distance(O, Epi.S'), distance(O, Epi.C) + distance(Epi.C,
            Epi.S')) by lemma segment addition — justified
      ✓ a3: equal(distance(O, Ecc.S), distance(O, Epi.S')) from a1, a2, d3, d4,
            d8, d9, h1, h2 by algebra — by calculation
      ✓ a4: parallel(line(O, Ecc.S), line(O, Ecc.E)) from apogee by lemma one
            line — justified
      ✓ a5: collinear(O, Ecc.E, Epi.S') from def Epi.C, def Epi.S', def Ecc.E by
            construction — read off the construction
      ✓ a6: parallel(line(O, Epi.S'), line(O, Ecc.E)) from a5 by lemma one line
            — justified
      ✓ a7: coincident(Ecc.S, Epi.S') from a4, a6, a3 by lemma same ray —
            justified
    ✓ perigee: coincident(Ecc.S, Epi.S') from p7 — justified (it is p7)
      ✓ p1: equal(distance(Ecc.E, Ecc.S), distance(Ecc.E, O) + distance(O,
            Ecc.S)) by lemma segment addition — justified
      ✓ p2: equal(distance(Epi.C, O), distance(Epi.C, Epi.S') + distance(Epi.S',
            O)) by lemma segment addition — justified
      ✓ p3: equal(distance(O, Ecc.S), distance(O, Epi.S')) from p1, p2, d3, d4,
            d8, d9, h1, h2 by algebra — by calculation
      ✓ p4: parallel(line(O, Ecc.S), line(O, Ecc.E)) from perigee by lemma one
            line — justified
      ✓ p5: collinear(O, Ecc.E, Epi.S') from def Epi.C, def Epi.S', def Ecc.E by
            construction — read off the construction
      ✓ p6: parallel(line(O, Epi.S'), line(O, Ecc.E)) from p5 by lemma one line
            — justified
      ✓ p7: coincident(Ecc.S, Epi.S') from p4, p6, p3 by lemma same ray —
            justified
  ✓ conclusion: coincident(Ecc.S, Epi.S') from same — justified (it is same)
```

Listing 8 The same derivation with one line altered in each case, and what the checks report.

```
# the wrong proposition is cited
✗ d6: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) from sides, d5 by Elements
      I.29
      — true, not justified (the proposition is not a conclusion of the rule)
      level reached: true

# a hypothesis of the proposition is not cited
✗ d6: parallel(line(O, Ecc.E), line(Epi.C, Ecc.S)) from sides by Elements I.33
```

```
      — true, not justified (a hypothesis of the rule is not among the cited
            premises)

# a false proposition
✗ d12: coincident(Ecc.S, Epi.C) from joins, d11, d10 by lemma same ray
      — false (residual 0.462 of the scale at kappa_bar = 10° for Ecc.R = 224.2,
            Ecc.e = 103.5, lamA = 80.62, T = 372.7, kappa_0 = 33.21, Epi.R' =
            224.2, Epi.r = 103.5)
      level reached: grounded

# a hypothesis is used without being cited
✗ conclusive: d10 holds only under h2, which is not cited on the way to it (a
      tacit premise)

# a proposition that needs a hypothesis is passed off as read off the
      construction
✗ d5: equal(distance(Ecc.E, Ecc.S), distance(O, Epi.C)) from def Ecc.S, def
      Epi.C by construction
      — true, not justified (it does not hold without the hypotheses (residual
            0.899 of the scale at kappa_bar = 15° for Ecc.R = 59.08, Ecc.e =
            21.37, lamA = 192, T = 393.8, kappa_0 = 119.4, Epi.R' = 585.3, Epi.r
            = 266.6), so it is not read off the constructions alone)

# a case is left out: the perigee
✗ justified: same: coincident(Ecc.S, Epi.S') by cases: no case covers kappa_bar
      = 180°
      level reached: true
```

Table 8 The rules of derivation (`spec/rules.yaml`). The wording of each proposition was compared with the page of Heath's translation in the Famulus collection; the lemmas are the author's.

rule	kind	case stated	Heath 1908
Elements I.13	proposition	a straight line set up on a straight line makes angles equal to two right angles	vol. I, p. 275
Elements I.28	proposition	exterior angle equal to the interior and opposite angle: the lines are parallel	vol. I, p. 309
Elements I.29	proposition	parallels make equal alternate angles	vol. I, p. 311
Elements I.30	proposition	lines parallel to one line are parallel	vol. I, p. 314
Elements I.32	proposition	the exterior angle of a triangle	vol. I, p. 316
Elements I.33	proposition	the lines joining equal and parallel lines	vol. I, p. 322
Elements I.34	proposition	opposite sides of a parallelogram	vol. I, p. 323

rule	kind	case stated	Heath 1908
Elements I.47	proposition	the square on the hypotenuse	vol. I, p. 349
Elements VI.6	proposition	one equal angle between proportional sides: the triangles are equiangular	vol. II, p. 204
lemma same ray	lemma	equal distances from a point along parallels to one line, the same way: one point	rests on I.30 and I.3
lemma same angle	lemma	equal angles with a ray on the same side: one direction	rests on I.23
lemma segment addition	lemma	a point between two others: the whole is the sum of the parts	rests on Common Notion 4
lemma one line	lemma	three points on one line: the lines through them are one	rests on Definition I.23
trigonometry sine, cosine, tangent	definition	the right triangle	—
common notion 1	common notion	things equal to the same thing are equal	vol. I, p. 155
common notion 5	common notion	the whole is greater than the part	vol. I, p. 155
construction, algebra, expansion, cases	—	the other licences	—

7.3 What the figure supplies

Euclid's propositions take something from the figure that their enunciations do not say: that points are distinct, that two points lie on the same side of a line, that two segments point the same way. In the rule table these are **configuration conditions**. I.33 has two: the equal and parallel lines point the same way, and three of the four points are not on one line. I.28 needs the two points on the same side of the transversal. The conditions are evaluated in every sampled configuration. A state in which a condition fails is a **degenerate state** of the step: the rule does not apply there.

For the argument of Fig. 1.2 this matters. At the apsides, $\bar{\kappa} = 0°$ and $\bar{\kappa} = 180°$, the points O, E, S and C lie on the apsidal line. There is no parallelogram, I.28 has no side to speak of, and I.33 has no figure. The conclusion is true there as well: the sun of both forms is at the apogee, or at the perigee. But the argument by the parallelogram does not reach it, and the printed figure, drawn at $\bar{\kappa} \approx 41°$, does not show that it does not (Figure 5). An earlier version of the derivation (paper version 1.2) had the parallelogram steps alone, and the checks reported the two states as degenerate. The derivation now distinguishes three cases. The first has the condition that O, E and S are not on one line; the other two argue with lengths along the apsidal line. With them no state is left uncovered (Listing 7), and leaving one out is reported (Listing 8).

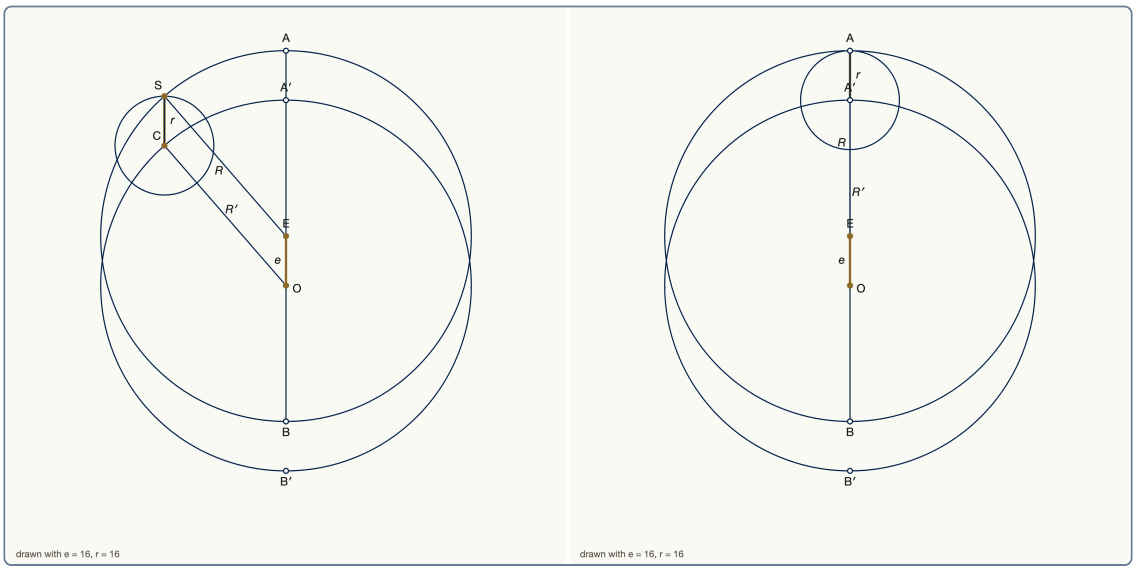


Figure 5 Left, the parallelogram OESC at κ̄ = 41°, with OE and CS in gold (step d6). Right, the same at κ̄ = 0°: the four points lie on the apsidal line, and Elements I.33 has nothing to apply to.

7.4 The proportional case

Under the hypothesis R′/r = R/e the two suns do not coincide, and the argument is another one. The triangles OES and S′CO have an equal angle at E and at C, both the supplement of κ̄, between proportional sides, e/R = r/R′. So they are equiangular (Elements VI.6: “if two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular”), and the angle at O equals the angle at S′. That angle is the alternate angle of S′OE, since CS′ is parallel to OE (I.29). So S and S′ make the same angle with OE on the same side, and are seen from O in the same direction. Eleven steps and the conclusion: six are read off the constructions, one is a calculation, the others are instances of rules or of Common Notion 1. The conclusion is `equivalent(Ecc, Epi, direction)`. The same two states are degenerate.

7.5 A derivation with an auxiliary construction

The page prints, without derivation, the equation of centre as a function of the mean anomaly: tan c = e sin κ̄ / (R + e cos κ̄), and its sine form (p. 22). Listing 9 derives both in the eccentric model alone. The derivation is supplied by the author of the astra text, and its header says so; it is not attributed to the witness.

Listing 9 The derivation `solar-equation-of-centre`. `construct` adds the foot F of the perpendicular from the earth on ES to the context. Two `compute` operations compare the largest equation with the values the page reports.

```
derivation solar-equation-of-centre in solar-eccentric "The equation of centre
      is the angle at the sun, with tan c = e sin κ̄/ (R + e cos κ)"
witness fr:kepler-sec/Swerdlow2026/0052/p03 "the formula, p. 22"
witness fr:kepler-sec/Swerdlow2026/0052/p05 "the maximum, p. 22"
author "Gerd Graßhoff"
construct F = foot(O, line(E, S)) "foot of the perpendicular from the earth on
      ES"
derive d1: equal(angle(S, E, A), angle(E, S, O) + angle(S, O, E)) from def A by
      Elements I.32 "OE produced to the apogee: the exterior angle at E of the
      triangle OES"
derive d2: equal(abs(norm180(c)), angle(O, S, E)) from d1, prop p1, prop p2 by
      algebra "the exterior angle is κ̄and the angle at O is κ, so the angle at
      S is κ̄− κ = c"
derive d3: perpendicular(line(O, F), line(E, S)) from def F by construction
derive d4: equal(distance(O, F), distance(O, E) * sin(angle(O, E, F))) from d3
      by trigonometry sine
```

```
derive d5: equal(distance(O, E), e) from def E by construction
derive d6: equal(sin(angle(O, E, F)), abs(sin(kappa_bar))) from def E, def S,
      def F by construction "the angle OEF is κ̄ or its supplement, according to
      the side of E on which F falls"
derive d7: equal(distance(O, F), e * abs(sin(kappa_bar))) from d4, d5, d6 by
      algebra
derive d8: equal(distance(S, F), R + e * cos(kappa_bar)) from def E, def S, def
      F by construction "F lies on SE produced beyond E for κ̄ within 90° of the
      apogee, between S and E otherwise: SF = SE ± EF"
derive d9: equal(tan(angle(O, S, F)), distance(O, F) / distance(S, F)) from d3
      by trigonometry tangent
derive d10: equal(angle(O, S, F), angle(O, S, E)) from def F, def E, def S by
      construction "F and E lie from S on one ray"
derive d11: equal(distance(O, F), distance(O, S) * sin(angle(O, S, F))) from d3
      by trigonometry sine
derive d12: equal(distance(O, S) * distance(O, S), distance(O, F) * distance(O,
      F) + distance(S, F) * distance(S, F)) from d3 by Elements I.47
conclude equal(tan(abs(norm180(c))), e * abs(sin(kappa_bar)) / (R + e *
      cos(kappa_bar))) from d2, d9, d10, d7, d8 by algebra "tan c = e sin κ̄ /
      (R + e cos κ), in absolute value"
conclude equal(sin(abs(norm180(c))), e * abs(sin(kappa_bar)) / sqrt(R * R + e *
      e + 2 * R * e * cos(kappa_bar))) from d2, d10, d11, d12, d7, d8 by algebra
      "sin c = e sin κ̄ / (R² + e² + 2Re cos κ)^½, in absolute value"
compute c_max = max(angle(O, S, E)) with solar-eccentric.ptolemy reported 2;23°
      source fr:kepler-sec/Swerdlow2026/0052/p05 "Ptolemy's maximum equation"
compute c_max_battani = max(angle(O, S, E)) with solar-eccentric.battani
      reported 1;59,10° source fr:kepler-sec/Swerdlow2026/0052/p06 "al-Battānī's
      maximum equation"
```

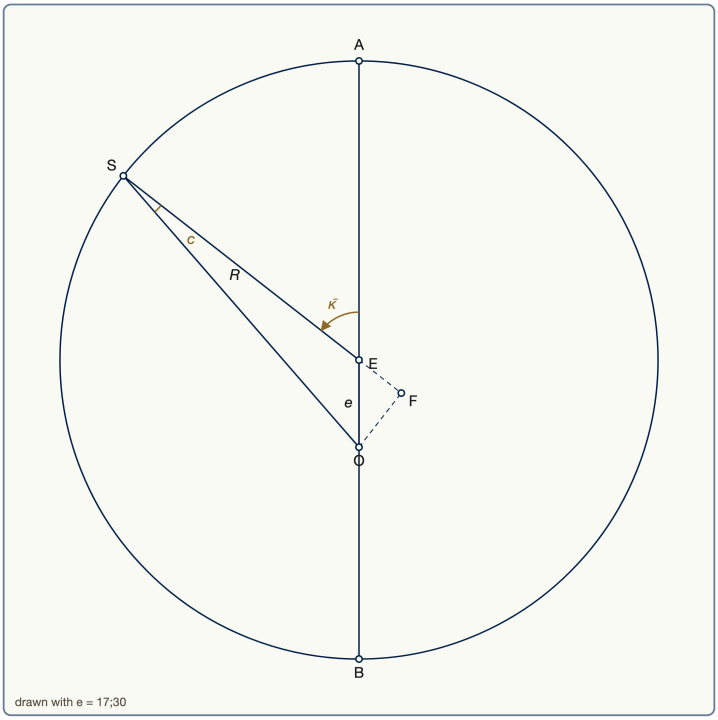


Figure 6 The scene of the derivation: the perpendicular OF from the earth on the line ES. The equation of centre c is the angle at the sun.

The auxiliary point F does not belong to the model. It exists in the derivation's context only (Figure 6). The steps use the exterior angle of the triangle OES (I.32), the right triangles OFE and OFS (the definitions of sine, cosine and tangent, and I.47 for OS), and three steps read off the construction. One of these, that SF = R + e cos $\bar{\kappa}$, hides a distinction of cases — F falls beyond E or between S and E — and is the weakest step: it is confirmed over the parameter space and not derived. The report shows this by its grade. Four states are degenerate: at 0° and 180° the triangle OES vanishes, at 90° and 270° F falls on E.

`compute` evaluates a quantity under a parametrisation and compares it with a reported value at the precision in which that value is written. The largest equation under Ptolemy's values is 2;23°, as printed. Under al-Battānī's values it is 1;59,9°; the page reports 1;59,10°.

7.6 Parameters determined from observations

The page reports how Ptolemy found the two parameters: from the vernal equinox to the summer solstice 94½ days "with mean motion 93;9°", from the summer solstice to the autumnal equinox 92½ days "with mean motion 91;11°", "from which he finds that where R = 60, e = 2;29,30 ≈ 2;30 … and λ_A = 65;30°" (p. 22). The operation `determine` inverts the model numerically for the unknown parameters, given the time or the arc of mean motion in which an observable goes from one value to another, and reports whether the solution it finds is the only one (Listing 10).

Listing 10 Eccentricity and apogee determined through the model, from the day counts and from the reported arcs.

```
$ famulus-astra derive solar-parameters-from-seasons
  = days.e: 2;28,58 at the only solution found; days turned into arcs of mean
        motion with T = 365;14,48 d; reported 2;29,30: discrepancy, difference
        −0;0,31,44
  = days.lamA: 65;26° at the only solution found; days turned into arcs of mean
        motion with T = 365;14,48 d; reported 65;30°: discrepancy, difference
        −0;3,47
  = arcs.e: 2;29,28 at the only solution found; the intervals taken as arcs of
        mean motion; reported 2;29,30: discrepancy, difference −0;0,1,51
  = arcs.lamA: 65;35° at the only solution found; the intervals taken as arcs of
        mean motion; reported 65;30°: discrepancy, difference +0;5,7
```

From the two reported arcs the model gives e = 2;29,28 and λ_A = 65;35°. From the day counts, turned into mean motion with the year of 365;14,48 days and exact arithmetic, it gives e = 2;28,58 and λ_A = 65;26°. Ptolemy's 2;29,30 and 65;30° agree with the first pair to within what a computation with a table of chords loses, and not with the second. The reported parameters follow from the rounded arcs. Swerdlow's paragraph says of this derivation that it "is so sensitive to errors in the intervals of time … that the exact results are fortuitous"; the two determinations show the sensitivity within Ptolemy's own numbers. Ptolemy's steps are not on the page, and Almagest III 4 is not among the texts available to us in addressable form: a gap.

7.7 What a derivation is and is not

A derivation that passes the four checks is reasoning whose every proposition is true in the model and whose every step names a rule that licenses it. It is not a formal proof in an axiomatic system. Truth is tested on samples. Rules are matched as schemas, and the schema states one case of a proposition. Configuration conditions are evaluated numerically. What the checks give is what a careful reader of a geometrical argument wants to know: which proposition licenses this step, are its hypotheses all there, where does it not apply, and does the conclusion use something that was never said.

8 What follows from the texts

8.1 Sheets

Every text has a specification sheet that restates it as tables: for a model the parameter symbols, the kinematic variable, the construction, the observables and the stated propositions with their check; for a parametrisation the values with their sources (Listing 11); for a derivation the givens, the steps with premises, rule, status, binding, configuration conditions and degenerate states, the rules with Heath's wording and page, and the computed quantities. A sheet does not mix kinds: a model's sheet has no values, and names its parametrisations, figures and derivations by address.

Listing 11 The YAML sheet of the parametrisation `fr:astra/solar-eccentric.ptolemy`, abridged to its head, its first two values and its gaps.

```
address: fr:astra/solar-eccentric.ptolemy
kind: parametrisation
title: Ptolemy's values
attributed: Ptolemy
witness:
- address: fr:kepler-sec/Swerdlow2026/0052/p05
  note: ''
author: Gerd Graßhoff
of: fr:astra/solar-eccentric
check:
  level: attested
  complete: true
values:
- symbol: R
  display: R
  meaning: radius of the eccentric
  as_written: '60'
  value: 60.0
  sexagesimal: '60'
  unit: length
  note: the norm of the radius
  source: fr:kepler-sec/Swerdlow2026/0052/p05
  gap: null
  taken_from: null
- symbol: e
  display: e
  meaning: eccentricity, the distance OE
  as_written: 2;30
  value: 2.5
  sexagesimal: 2;30
  unit: length
  note: 2;29,30 rounded to 2;30
  source: fr:kepler-sec/Swerdlow2026/0052/p05
  gap: null
  taken_from: null
gaps:
- 'kappa_0 = 0°: the pages give no epoch; t is counted from a passage through
      the apogee'
…
```

8.2 Figures and animation

A figure text says what it draws (Listing 12). Fig. 1.2 and Fig. 1.4 are two figure texts; the second draws the eccentric model alone, under the same values, at $\bar{\kappa} = 108°$, with the solstitial colure upright (Figure 7).

Drawing values replace attested ones for legibility; they stand in the figure text, appear in the corner of the figure and never touch a parametrisation.

Listing 12 The figure text for Swerdlow's Fig. 1.2.

```
figure solar-fig-1-2 of solar-eccentric-epicyclic "The two forms of Ptolemy's
      solar model superposed, R' = R and r = e"
witness fr:kepler-sec/Swerdlow2026/0050/fig01
author "Gerd Graßhoff"
with solar-eccentric-epicyclic.equal-radii
drawn with Ecc.e = 16, Epi.r = 16
at kappa_bar = 41°
up lamA
fit [Ecc.ecc, Epi.def, Epi.epi]
speed 24 s
display Epi.S' "S'"
label angle kappa_bar at Ecc.E from Ecc.A to Ecc.S arrow
label angle kappa_bar at O from Epi.A' to Epi.C arrow
label length Ecc.R on Ecc.E Ecc.S
label length Ecc.e on O Ecc.E
label length Epi.R' on O Epi.C side right
label length Epi.r on Epi.C Epi.S' side right
label segment line(Ecc.A, Ecc.B)
label arrow Ecc.S on Ecc.ecc
label arrow Epi.C on Epi.def
label dashed line(O, Epi.S')
```

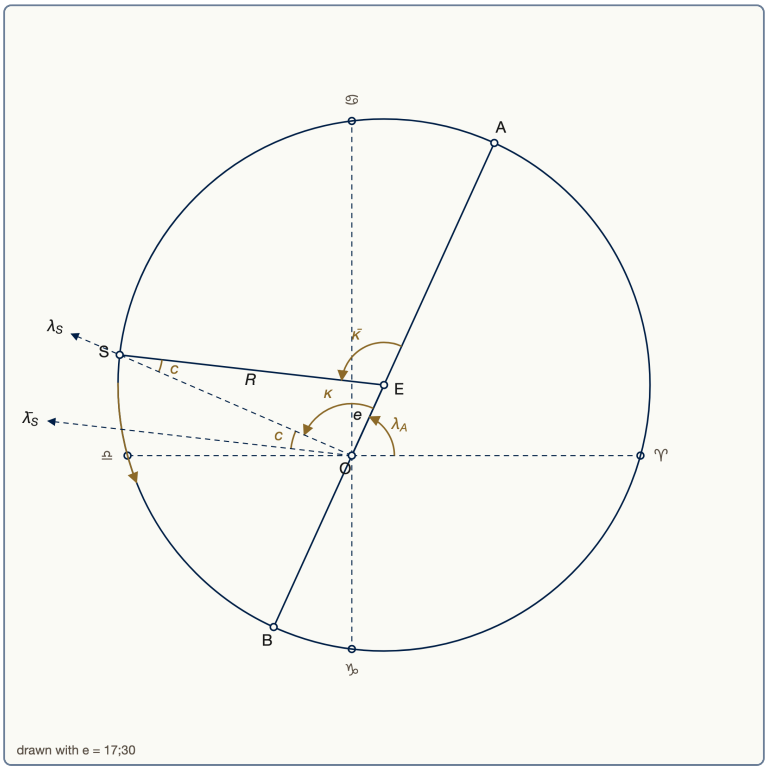


Figure 7 Swerdlow's Fig. 1.4 drawn from the figure text `solar-fig-1-4`: the eccentric model under Ptolemy's values at $\bar{\kappa} = 108°$, with the longitudes λ_S and $\bar{\lambda}_S$ as directions from O, the angles λ_A, $\bar{\kappa}$ and κ, the equation of centre c at S and at O, and the cardinal points of the zodiac. It is the model of Figure 2 in another state and another view.

The animation is a page with the figure in its depicted state, a bundle — the context's operations, the figure's parameter values, the scene — and a runtime of 468 lines of JavaScript without libraries. The runtime computes configurations with the same operations as the Python evaluator and redraws the figure at every frame. It has one control per thing to do: a slider for the kinematic variable; for a superposition a slider that fades between the two members, which is Duke's demonstration of the equivalence [7]; for a figure of a derivation two buttons that go through the steps, each shown with its proposition, rule and status and with what it speaks of in gold.

Two evaluators of one operation set can drift apart. They are held together by shared cases: 64 written by hand, at least one per operation including the cases without solution, and the values of every named object of every context under sampled and attested parameter values in eight states, six given by the variable and two by a time. On 10 November 2026 all 3,914 values agreed to 10^{-9}. A headless browser photographs an animation at three moments and requires the frames to differ.

8.3 Reference objects

Famulus gives every object it keeps an address beginning with `fr:` [8]. Every astra text and every part of it has one: `fr:astra/solar-eccentric` is the model, `fr:astra/solar-eccentric.ptolemy/e` a value with its source, `fr:astra/solar-eccentric-epicyclic/Epi.C` the centre of the epicycle in the superposition, `fr:astra/solar-equivalence-equal-radii/d6` a step (Listing 13). Prose elsewhere can then say exactly what it speaks of.

Listing 13 The reference service answering for step d6.

```
{
 "ref": "fr:astra/solar-equivalence-equal-radii/d6",
 "kind": "astra_step",
 "found": true,
 "proposition": "parallel(line(O, Ecc.E), line(Epi.C, Ecc.S))",
 "from": [
  "sides",
  "d5"
 ],
 "by": "Elements I.33",
 "status": "justified",
 "binding": {
  "A": "O",
  "C": "Ecc.E",
  "B": "Epi.C",
  "D": "Ecc.S"
 },
 "hypotheses_met_by": {
  "parallel(line(A, B), line(C, D))": "sides",
  "equal(distance(A, B), distance(C, D))": "d5"
 },
 "configuration_conditions": [
  "same_direction(O, Epi.C, Ecc.E, Ecc.S)",
  "noncollinear(O, Epi.C, Ecc.E)"
 ],
 "degenerate_at": null,
 "in": "fr:astra/solar-equivalence-equal-radii"
}
```

8.4 Comparing

Two models are compared for identity, two parametrisations value by value (Listing 14). The comparison states and does not interpret. That two different models agree in what they show is not something it can say.

Listing 14 The two forms are different geometries that share only the observer. The two parametrisations differ in the eccentricity and in the apogee, the latter by 16;47°.

```
$ famulus-astra diff solar-eccentric solar-epicyclic
solar-eccentric ↔ solar-epicyclic: different geometries
  parameters
    − R
    − e
    + R'
    + r
  construction
    − E (point) = O + polar(e, lamA)
    − ecc (circle) = circle(E, R)
    − A (point) = E + polar(R, lamA)
    − B (point) = E - polar(R, lamA)
    − S (point) = E + polar(R, lamA + kappa_bar)
    + def (circle) = circle(O, R')
    + A' (point) = O + polar(R', lamA)
    + B' (point) = O - polar(R', lamA)
    + C (point) = O + polar(R', lamA + kappa_bar)
    + epi (circle) = circle(C, r)
    + S' (point) = C + polar(r, lamA)
  observables
    − lam_bar (angle) = direction(E, S)
    − kappa (angle) = lam - lamA
    − c (angle) = lam_bar - lam
    ~ lam: direction(O, S) → direction(O, S')
  shared construction: O

$ famulus-astra diff solar-eccentric.ptolemy solar-eccentric.battani
solar-eccentric.ptolemy ↔ solar-eccentric.battani
  values of Ptolemy ↔ al-Battānī
    R: 60 ↔ 60  (equal)
    e: 2;30 ↔ 2;4,45  (−0;25,15)
    lamA: 65;30° ↔ 82;17°  (+16;47)
    T: 365;14,48 ↔ —
    kappa_0: 0° ↔ —
```

9 Case study: Ptolemy's solar model in Swerdlow's exposition

Table 9 lists the eighteen texts. All but one pass every level of their check. The exception is al-Battānī's parametrisation, which is incomplete because the page is.

Table 9 The texts of the case study (`examples/solar/`).

text	kind	what it is
`solar-eccentric`, `solar-epicyclic`	model	the two forms; 54 and 40 operations
`solar-eccentric.ptolemy`, `.battani`	parametrisation	Ptolemy's values; al-Battānī's (partial)
`solar-eccentric-epicyclic`	superposition	the scene of Figs. 1.2 and 1.3; 84 operations
`…epicyclic.equal-radii`, `.proportional`	parametrisation	Ptolemy's values with the radii of Fig. 1.2 and of Fig. 1.3

text	kind	what it is
`solar-equivalence-equal-radii`	derivation	2 givens, 26 steps in 3 cases; equivalence in position
`solar-equivalence-proportional`	derivation	1 given, 11 steps; equivalence in direction
`solar-equation-of-centre`	derivation	1 auxiliary construction, 12 steps, 2 conclusions, 2 computations
`solar-parameters-from-seasons`	derivation	2 determinations
`solar-fig-1-2`, `-1-3`, `-1-4` and four more	figure	the three printed figures; the members alone; the scenes of two derivations

Writing the texts was a reading of the pages. It found the following.

The parallelogram argument does not cover the apsides. Steps by I.28 and I.33 have configuration conditions that fail at $\bar{\kappa} = 0°$ and 180°. The page's sentence "complete parallelogram OESC" presupposes a position in which there is one. The complete argument has three cases, of which page and figure give one.

The reported parameters follow from the rounded arcs. Inverting the model for the reported arcs gives Ptolemy's eccentricity to two units of the second sexagesimal place and his apogee to 0;5°; inverting it for the day counts does not.

A misprinted sign rule and a misprinted difference. On page 22 the rule for the sign of c reads "for $\bar{\kappa} > 180°$" in both clauses; the model gives $c > 0°$ for $\bar{\kappa} < 180°$. The same page gives the advance of al-Battānī's apogee over Ptolemy's as 16;37°; the two longitudes on the page differ by 16;47°, as the comparison of the two parametrisations shows (Listing 14). Both are treated in a companion paper that reads these pages assertion by assertion [9].

Errors of our own. The first draft gave the year the locus of page 19, which prints the mean motion of the Alfonsine Tables; the Ptolemaic year stands on page 17. A source being named is not a source being right, and the address makes the check a quick one. The second version (1.1) of this paper made the four conflations listed in Section 1. And a figure in the companion paper, al-Battānī's largest equation recomputed as 1;59,8°, was truncated where it should have been rounded: it is 1;59,9°.

10 Representation, model, description

The **model** is not a text. It is the geometry: the function from parameter values and states to configurations, with what holds of it. Ptolemy's eccentric solar model is the same model whether Ptolemy, Regiomontanus, Swerdlow or an astra text states it. That is why a model text names three persons or texts and not one.

A **representation** is anything that determines the model: Ptolemy's prose and figure, Swerdlow's prose and figures, the astra model text, the operation list, the sheet. Two representations represent the same model when they agree in every configuration. A **figure** represents less and more than the model. Less, because it shows one state under one set of values. More, because it chooses a view, and often values that are not the attested ones. Figs. 1.2 and 1.4 draw one model in two states, two orientations and with an exaggerated eccentricity.

A **diagram of two models** is a superposition. Fig. 1.2 is drawn so that a parallelogram can be seen, and the parallelogram is the place of an argument. Three things are to be told apart: the superposition, which the diagram shows; the hypotheses under which something holds across the two models, under which the diagram is drawn and which it cannot show; and the derivation, whose scene the diagram is. The equivalence

of the two models is the conclusion of the third. It is a relation between models, not between representations of one model.

The **describing text** refers to the model through names — "complete parallelogram OESC", "r is always parallel to the apsidal line AB" — and asserts things of different kinds. The astra texts make the names resolvable and each assertion checkable as what it is (Table 10).

Table 10 Kinds of assertion in a describing text, and how each is checked.

the prose	is checked as
states a property of the construction	a proposition of the model
says that something follows	a step of a derivation, with its rule
gives a number that follows from other numbers	arithmetic, or a quantity computed under a parametrisation
gives parameters found from observations	a determination through the model

11 Related work

Languages for geometrical constructions are older than dynamic geometry on screens. **GCLC**, the Geometry Constructions Language of Janičić [2], treats a construction as a procedure and has a built-in prover that takes a conjecture about the construction and returns a proof. astra/1 takes from GCLC the view of a construction as a procedure whose objects depend only on earlier ones. Its derivations do something different from GCLC's prover: they take the steps that a historical text gives or presupposes, with the propositions that text would cite, and ask of each step whether it is true and whether that proposition licenses it. The aim is to analyse a given piece of reasoning, not to find a proof. **Eukleides** [10] makes the text resemble geometric notation, works with typed values and avoids Cartesian coordinates where it can; astra/1 is typed with units and has no coordinates in its texts. **Cinderella** [11], **GeoGebra** [12] and **JSXGraph** [13] made constructions interactive: a free element or a slider drives a graph of dependent objects. In such a construction, values and geometry are stored together; astra/1 separates them.

Animations of historical models are a separate tradition. Dennis Duke's animations of the Almagest models [7] compute every position in closed form from one time variable and demonstrate the equivalence of eccentric and epicyclic forms by a slider that turns one into the other. Brandon Rhodes' scale drawings of Ptolemy's cosmos [14] fit their parameters to a modern ephemeris and state their scale conventions. astra/1 shares the fade between two models with Duke and the stated conventions with Rhodes, and adds what neither aims at: the geometry apart from the values, a source for each number, the equivalence as a derivation with hypotheses, and an address for every part. The Famulus astronomy expert [15] computes positions from the Almagest models; astra/1 describes the constructions those computations presuppose.

Table 11 summarises the differences. It records design aims, not quality: the geometry packages are general-purpose tools, astra/1 is a language for one kind of object.

Table 11 Design aims of astra/1 and the work it learned from.

	GCLC	Eukleides	Cinderella, GeoGebra, JSXGraph	Duke	astra/1
a construction is a text	yes	yes	stored files	code	yes
coordinates	yes	avoided	yes	yes	only the observer
units typed	no	types, no units	no	no	yes

	GCLC	Eukleides	Cinderella, GeoGebra, JSXGraph	Duke	astra/1
geometry apart from values	no	no	no	no	model and parametrisation
two models in one scene	no	no	by hand	slider	superposition
reasoning	prover finds a proof	no	no	no	a given derivation is checked step by step
sources	no	no	no	no	required or gap
citable parts	no	no	no	no	`fr:astra/…`

12 Limitations

Cases have to be found by the author of a derivation: the checks report the states a step does not cover and whether the cases cover every configuration, but they do not propose the distinction. The cases hidden in a step that is read off the construction are not shown. Hypotheses have to fix a parameter; inequalities between parameters and hypotheses of incidence are not supported. The grades *read off the construction* and *calculation* are tested numerically and not replayed symbolically; an expansion is tested at three sizes of the small ratio and not derived term by term. A rule schema states one case of a proposition, and the rule table holds the rules that the derivations written so far needed. Truth is tested on samples throughout.

The language is planar; latitude needs a third dimension. Motions taken from tables are not implemented. Only the solar model is written. The language covers the equant (`seen_from`), the Tusi couple (`rotate`) and Kepler's ellipse (`true_anomaly`, `ellipse`), but no such model is written, because the rule that every value has a source that can be addressed is kept: the later chapters of Swerdlow, which announce "more interesting transformations of planetary models based upon the same principles that lead from the Ptolemaic to the Copernican and Tychonic systems" (p. 20), are not yet read for loci. They are superpositions and derivations waiting to be written.

The level *attested* checks that a source is named, not that it says what the parametrisation says. The propositions of Euclid are cited by the page of Heath's translation, not by paragraph, because those volumes have no paragraph addressing yet. The Almagest itself is not among the addressable texts, so the attribution to Ptolemy and the loci in the Almagest are as Swerdlow's page gives them. The code is not yet public.

13 Conclusion

A historical astronomical model can be written so that four things stay apart: the geometry, as a model without values; the numbers, as parametrisations with owners and sources; the position, as a state; and the reasoning, as a derivation. A diagram that shows two models is then a superposition, and what it is drawn to make visible, an equivalence, is the conclusion of a derivation under hypotheses the diagram cannot show. A derivation whose steps name premises and rules can be checked step by step: for truth over the parameter space, for whether the step is an instance of its rule, for the states in which the rule does not apply, and for premises used without being said. Applied to Ptolemy's solar model as Swerdlow uses it, this showed that the parallelogram argument needs two further cases at the apsides and that the reported parameters follow from rounded arcs. Neither is visible in the diagram.

Availability

The language specification (`spec/astra-1.md`), the operation set, the rules of derivation, the test cases, the eighteen texts, the runtime, the deep-reading tool and the tests are in the Famulus workspace `famulus-astronomica` (41 tests; 6,467 lines of Python and JavaScript, of which 952 for derivations and rules, 468 for the runtime and 354 for the deep-reading tool). The addresses `fr:astra/…` resolve on the Famulus reference service of the author's node; the names used in version 1.1 of this paper resolve to the objects they have become. Swerdlow's book is cited by its scan pages in the Famulus collection `kepler-sec` (scan page n is printed page n − 30 in chapter 1); Heath's Euclid by the scan pages of the collection `assyrischeSammlung`.

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