Euler's Argument for the Tide-Raising Force, Reconstructed

One Segment of a Source, Passage by Passage, beside Its Formal Reconstruction

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Abstract

One segment of a historical source is taken through its argument: the end of §22 and §§24 to 29 of Euler's *Inquisitio physica in causam fluxus ac refluxus maris* (Eneström 57), seven pages of Opera omnia II 31, in which Euler derives the force by which the sun or the moon tends to raise the tides. The segment is divided into twelve passages. Each passage is quoted in Latin and in English and set beside the step of the reconstruction that says the same thing explicitly, with its premises, the rule that licenses it, and the result of a check. Annotations stand in a third column. The texts are not typed: the Latin is taken from the page of the edition, the English from the edition's stored translation, and every passage links to its page in the edition. The reconstruction confirms Euler's exact components and his expansion. It follows the first edition and the manuscript against an emendation of the editor, which a test shows to be wrong. It marks where it does not follow Euler's route: the resolution by similar triangles in §26 is compared with the result, not rebuilt. The format, called an exposition, is general.

keywords history of astronomy · Euler · tides · geometrical reasoning · argument reconstruction · source edition

1 The segment

In chapter II of his prize essay on the tides [1] Euler asks by what force the sun or the moon tends to move a particle of the earth relative to the earth's centre. He answers on seven pages, 33 to 39 of the volume of the Opera omnia: a principle at the end of §22, a figure in §24, the force resolved exactly in §§25 and 26, expanded in a small ratio in §27, and read for its consequences in §§28 and 29. An earlier paper treated this argument briefly as one of two cases [2]. Here it is the only subject, and the aim is different: to set Euler's own words beside the reconstruction, passage by passage, so that a reader can see for every sentence what has been made of it.

The method is that of the series [3], [4]. The *model* is written out: the points of Euler's figure with their constructions, the lengths as symbols, the forces as quantities defined on them. The *argument* is written as steps, one proposition each, with the premises it is drawn from and the rule that licenses it. A program *checks* whether every premise is there, whether every proposition holds in the model for every admissible value and position, whether every step is an instance of its rule, and whether the conclusion rests only on what is cited.

2 How to read the passages

The segment is divided into twelve passages. A passage is a stretch of Euler's text that does one thing: states the principle, sets up the figure, gives a formula, draws a consequence. Each passage has three parts.

The source. Euler's text, in Latin as the edition transcribes it, or in English. A mark such as |35| shows where a page of the Opera omnia begins. The English is the edition's stored page translation, a machine translation; it was compared with the Latin for this paper and is a working translation, not a published one. Three wordings were corrected, all three at a page break, where a translation made page by page had broken a sentence in two.

The reconstruction. The step or steps of the argument that say what the passage says. A step has a number, a claim in words, the proposition in the notation of the figure, the steps or constructions it is drawn from, the rule, and a mark: ✓ an instance of the rule named, ◦ read off the construction, ≈ by calculation or by expansion. Where a passage is not a step, the reconstruction says what it is instead: a definition in the model, or a part of the text that is not reconstructed.

The notes. What the reconstruction supplies that Euler leaves unsaid, where it departs from his route, and what the check found.

In print the source stands in two columns, Latin and English, with the reconstruction and the notes below. The HTML version of this paper shows the same passages as an interactive page: the source beside the reconstruction, the notes as a third column that can be switched on, a figure that shows what a selected step speaks of, and for every passage a link that opens the page of the Opera omnia in the edition's viewer.

3 The passages

§22 The principle: take away the force on the centre

Opera omnia II 31, p. 33

EULER

Cum autem iste motus, qui in singulis Terrae partibus generatur, a differentia inter vires centrum Terrae et ipsas partes sollicitantes proficiscatur, propria vis, qua quaeque particula agitabitur, innotescet, si a vi acceleratrice illam particulam sollicitante auferatur vis acceleratrix, quam centrum Terrae patitur: haecque subtractio ita instituitur, ut cuique particulae praeter vim actu eam sollicitantem alia vis aequalis illi, quam centrum patitur, in directione contraria applicata concipiatur: tum enim vis quae ex compositione harum duarum oritur, erit vera vis particulam illam de loco suo deflectens.

ENGLISH

Since, however, this motion, which is generated in the individual parts of the Earth, arises from the difference between the forces acting upon the center of the Earth and those acting upon the parts themselves, the proper force by which each particle is agitated will become known if the accelerating force which the center of the Earth experiences is subtracted from the accelerating force acting upon that particle: and this subtraction is performed in such a way that, in addition to the force actually acting upon each particle, another force equal to that which the center experiences, applied in the opposite direction, is conceived; for then the force which arises from the composition of these two will be the true force deflecting that particle from its place.

RECONSTRUCTION

Not a step of the argument but what the argument is about. In the model it is the definition of the two quantities observed: the force on the particle less the force on the centre, taken along MC and along MR.

tMC the force on the particle less the force on the centre, along MC, towards the centre — *defined in the model*
fM * cos(direction(M, S) - direction(M, C)) - fC * cos(direction(C, S) - direction(M, C))

tMR the same along MR, normal to MC, towards the side of the luminary — *defined in the model*
fM * cos(direction(M, S) - (direction(M, C) + 90°)) - fC * cos(direction(C, S) - (direction(M, C) + 90°))

NOTES

Everything that follows computes this difference. A particle is moved relative to the earth only by what the luminary does to it over and above what it does to the centre.

§24 The figure: the particle M by CP = x and PM = y

Opera omnia II 31, p. 34–35

EULER

24. Concipiatur nunc Terra tota tanquam globus $ADBE$ [fig. 3] urgeri ad Solem Lunamve in S existentem cuius vis absoluta seu ea, quam in distantia a |35| centro suo S semidiametro Terrae aequali exerit, sit $= S$, distantia vero centri Terrae C ab S seu CS ponatur $= a$; eritque vis acceleratrix, qua tota Terra tanquam in C collecta sollicitabitur in directione CS, $= \frac{S}{aa}$. Contemplemur iam particulam Terrae quamcunque M cuius situs ita sit definitus, ut sit $CP = x$ et $PM = y$, existente MP normali ad CS; hinc igitur habebitur $SP = a - x$ et $SM = \sqrt{((a-x)^2 + y^2)}$.

ENGLISH

24. Let the whole Earth now be conceived as a globe $ADBE$ [fig. 3] urged toward the Sun or Moon existing at S, whose absolute force, or that which it exerts at a distance from |35| its centre S equal to the Earth's semidiameter, be $= S$, and let the distance of the center of the Earth C from S, or CS, be set $= a$; then the accelerating force by which the whole Earth, as if concentrated at C, is urged in the direction CS, will be $= \frac{S}{aa}$. Let us now consider any particle of the Earth M whose position is defined such that $CP = x$ and $PM = y$, with MP being perpendicular to CS; hence we will have $SP = a - x$ and $SM = \sqrt{((a-x)^2 + y^2)}$.

RECONSTRUCTION

1 **§24: SM² = (a − x)² + y²** ✓
dist² = (a − x)·(a − x) + y² from 1.6

1.1 MP is normal to CS — *construction* ◦
MP ⊥ SC from constr. P

1.2	the right triangle MPS MS·MS = MP·MP + SP·SP from 1.1	*Elements I.47*	✓
1.3	SP = a − x from constr. S, constr. P, constr. x	*construction*	∘
1.4	PM = y MP = |y| from constr. M, constr. P, constr. y	*construction*	∘
1.5	dist = MS from constr. dist	*construction*	∘
1.6	dist² = (a − x)·(a − x) + y² from 1.2, 1.3, 1.4, 1.5	*algebra*	≈

NOTES

Euler's letters: C the centre of the earth, S the sun or the moon, M a particle, P the foot of the perpendicular from M on CS. The letter S names both the luminary and its absolute force; the formal text of the reconstruction writes the force as `Sf`.

Euler's Fig. 3 also carries the points N, Q, O, V and R, which §§24–26 use for the resolution of the force. The figure of the reconstruction draws C, S, A, B, D, E, M and P only.

The model takes the distance $CM = \rho$ as a parameter and the angle $ACM = \varphi$ as what varies, so that $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. Euler keeps x and y and sets $\sqrt{x^2+y^2} = 1$ only in §29.

§24 The two forces on the particle

Opera omnia II 31, p. 35

EULER

Vis igitur acceleratrix, qua particula M versus S pelletur, erit $= \frac{S}{(a-x)^2+y^2}$; a qua cum auferri debeat vis, qua tota Terra versus S nititur, concipienda est particulae M applicata vis $= \frac{S}{aa}$ in directione MN ipsi CS parallela et opposita; quae duae vires particulam M aeque afficient ac si universa Terra quiesceret vel uniformiter in directum moveretur, qui casus ab illo non differt. Ex his igitur ambabus viribus conatus innotescet, quo particula M a vi ad S directa de loco suo recedere annitetur: ad ipsum autem motum definiendum insuper vis gravitatis erit respicienda: et quia haec particula non est libera, sed quaquaversus materia terrestri circumdata, investigari oportet, quantum ista materia effectum viribus sollicitantibus concedat.

ENGLISH

Therefore, the accelerating force by which the particle M is driven toward S will be $= \frac{S}{(a-x)^2+y^2}$; since from this must be subtracted the force by which the whole Earth tends toward S, we must conceive a force $= \frac{S}{aa}$ applied to the particle M in the direction MN parallel and opposite to CS; these two forces will affect the particle M in the same way as if the entire Earth were at rest or moving uniformly in a straight line, which case does not differ from that one. From these two forces, therefore, the endeavor will become known by which the particle M strives to depart from its place due to the force directed toward S; moreover, to define the motion itself, the force of gravity must also be taken into account; and since this particle is not free, but surrounded on all sides by terrestrial matter, we must investigate how much effect this matter allows to the soliciting forces.

RECONSTRUCTION

The two forces enter the model as defined quantities: the force on the particle towards S, and the force on the centre towards S, which the principle takes away.

fM	accelerating force on the particle towards S Sf / (dist * dist)	*defined in the model*
fC	accelerating force on the centre towards S Sf / (a * a)	*defined in the model*

NOTES

The force on M is written as S over the square of SM. The passage gives no argument for this form.

Euler applies the second force to the particle in the direction MN, parallel and opposite to CS; the model subtracts the force on the centre. The two say the same.

§25 A first resolution, along CS and along MP

Opera omnia II 31, p. 35–36

EULER

25. Quoniam autem in hoc capite nobis nondum est propositum in ipsum effectum ab his viribus oriundum inquirere, sed tantum conatum evolvere atque explorare; diligentius perpendemus, cuiusmodi vires ex combinatione harum potentiarum particulam M sollicitantium resultent. Hunc in finem resolvatur vis MS in duas laterales, quarum alterius directio parallela sit ipsi CS, altera vero in MP cadat: ex quo reperietur vis illa particulam M in directione MQ urgens $= \frac{S(a-x)}{((a-x)^2+y^2)^{3/2}}$; altera vero vis in directione MP trahens $= \frac{Sy}{((a-x)^2+y^2)^{3/2}}$. Cum autem particula M insuper trahatur in directione MN vi $= \frac{S}{aa}$, tres istae vires a Sole Lunave in S existente reducentur ad duas, quarum altera in directione MQ urgens erit $= \frac{S(a-x)}{((a-x)^2+y^2)^{3/2}} - \frac{S}{a^2}$, altera vero directionem habens $MP = \frac{Sy}{((a-x)^2+y^2)^{3/2}}$. Quare si rectae MQ et MP his viribus proportionales capiantur, et rectangulum $MQOP$ compleatur exprimet diagonalis MO tam directionem quam quantitatem vis ex tribus praecedentibus ortae: erit |36| autem anguli OMP tangens $= \frac{a-x}{y} - \frac{((a-x)^2+y^2)^{3/2}}{a^2y}$; quo cognito, si fiat ut MP ad MO ita $\frac{Sy}{((a-x)^2+y^2)^{3/2}}$ ad quartam, haec ipsa quarta proportionalis erit vis particulam M in directione MO sollicitans, quae oritur a vi ad S tendente.

ENGLISH

25. Since, however, in this chapter it is not yet our purpose to inquire into the actual effect arising from these forces, but only to develop and explore the endeavor, we shall examine more carefully what kind of forces result from the combination of these powers

soliciting the particle M. To this end, let the force MS be resolved into two lateral forces, the direction of one of which is parallel to CS, while the other falls along MP; from this, that force urging the particle M in the direction MQ will be found to be $= \frac{S(a-x)}{((a-x)^2+y^2)^{3/2}}$, and the other force pulling in the direction MP will be $= \frac{Sy}{((a-x)^2+y^2)^{3/2}}$. But since the particle M is additionally pulled in the direction MN by a force $= \frac{S}{aa}$, these three forces from the Sun or Moon existing at S will be reduced to two, of which one urging in the direction MQ will be $= \frac{S(a-x)}{((a-x)^2+y^2)^{3/2}} - \frac{S}{a^2}$, and the other, having the direction MP, will be $= \frac{Sy}{((a-x)^2+y^2)^{3/2}}$. Therefore, if the straight lines MQ and MP are taken proportional to these forces, and the rectangle $MQOP$ is completed, the diagonal MO will express both the direction and the magnitude of the force arising from the three preceding ones: *|36|* and the tangent of the angle OMP will be $= \frac{a-x}{y} - \frac{((a-x)^2+y^2)^{3/2}}{a^2y}$; this being known, if it is made so that as MP is to MO, so is $\frac{Sy}{((a-x)^2+y^2)^{3/2}}$ to a fourth, this very fourth proportional will be the force urging the particle M in the direction MO, which arises from the force tending toward S.

RECONSTRUCTION

Not reconstructed. §26 replaces this resolution by the one along MC and MR, and no formula of §§26–29 cites the diagonal MO.

NOTES

The tangent is the quotient of the two components just found: $\frac{S(a-x)}{d^3} - \frac{S}{a^2}$ along MQ over $\frac{Sy}{d^3}$ along MP, with $d = SM$.

§26 The exact force towards the centre

Opera omnia II 31, p. 36

EULER

26. Ut autem istae vires facilius cum gravitate naturali, cuius directio est MC, coniungi queant, resolvantur eae in binas, quarum altera in ipsam directionem MC cadat, alterius vero directio sit MR normalis ad MC. Ad hoc commodissime praestandum, resolvatur vis MS primum in duas, quarum altera ut ante directionem habeat ipsi CS parallelam, alterius vero directio in ipsam MC incidat. Cum igitur sit $MC = \sqrt{(x^2+y^2)}$, erit prior vis $= \frac{Sa}{((a-x)^2+y^2)^{3/2}}$, posterior vero $= \frac{S\sqrt{(x^2+y^2)}}{((a-x)^2+y^2)^{3/2}}$, qua vis gravitatis augebitur. At si a priori auferatur vis $= \frac{S}{aa}$, remanebit vis particulam M in directione MQ sollicitans

$$= \frac{Sa}{((a-x)^2+y^2)^{3/2}} - \frac{S}{a^2}.$$

Iam ex Q in CM productam demittatur perpendiculum QV, eritque ob similitudinem triangulorum QVM et MPC vis gravitati contraria secundum directionem MV agens ex vi MQ orta

$$= \frac{Sax}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}} - \frac{Sx}{a^2\sqrt{(x^2+y^2)}};$$

unde omnino particula M a vi ad S tendente versus C urgebitur vi

$$= \frac{Sx}{a^2\sqrt{(x^2+y^2)}} - \frac{S(ax-xx-yy)}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}}.$$

ENGLISH

26. But in order that these forces may be more easily combined with natural gravity, whose direction is MC, let them be resolved into two, one of which falls along the direction MC itself, while the direction of the other is MR perpendicular to MC. To perform this most conveniently, let the force MS first be resolved into two, one of which, as before, has a direction parallel to CS itself, while the direction of the other falls along MC itself. Since, therefore, $MC = \sqrt{(x^2+y^2)}$, the former force will be $= \frac{Sa}{((a-x)^2+y^2)^{3/2}}$, and the latter indeed $= \frac{S\sqrt{(x^2+y^2)}}{((a-x)^2+y^2)^{3/2}}$, by which the force of gravity will be increased. But if from the former the force $= \frac{S}{aa}$ is subtracted, there will remain the force urging the particle M in the direction MQ

$$= \frac{Sa}{((a-x)^2+y^2)^{3/2}} - \frac{S}{a^2}.$$

Now let the perpendicular QV be dropped from Q onto CM produced, and because of the similarity of the triangles QVM and MPC, the force acting in the direction MV contrary to gravity, arising from the force MQ, will be

$$= \frac{Sax}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}} - \frac{Sx}{a^2\sqrt{(x^2+y^2)}};$$

whence, on the whole, the particle M will be urged toward C by the force tending toward S with a force

$$= \frac{Sx}{a^2\sqrt{(x^2+y^2)}} - \frac{S(ax-xx-yy)}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}}.$$

RECONSTRUCTION

2 **§26: towards the centre the particle is urged by Sx/(a²√(x²+y²)) − S(ax − xx − yy)/(((a−x)²+y²)^(3/2) √(x²+y²))** ✓

tMC = S·x / (a²·ρ) − S·(a·x − x² − y²) / (dist³·ρ) from 2.1

2.1 the force towards S resolved along CS and along MC, the force on the centre taken away, and what remains along MQ carried to MC by the similar triangles QVM and MPC *construction* ∘

tMC = S·x / (a²·ρ) − S·(a·x − x² − y²) / (dist³·ρ) from constr. tMC, constr. fM, constr. fC, constr. dist, constr. x, constr. y, constr. M, constr. S

NOTES

Euler reaches the expression through the perpendicular QV and the similar triangles QVM and MPC. The reconstruction does not rebuild that route. It compares Euler's closed expression with the definition of the quantity (the force on M less the force on C, taken along MC) in every sampled configuration; the two agree. This is the widest gap between the source and the reconstruction.

§26 The exact force across the vertical

Opera omnia II 31, p. 36

EULER

Praetera vero eadem particula M in directione MR ad MC normali sollicitabitur vi

$$= \frac{Say}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}} - \frac{Sy}{a^2\sqrt{(x^2+y^2)}}.$$

ENGLISH

Furthermore, the same particle M will be urged in the direction MR perpendicular to MC by the force

$$= \frac{Say}{((a-x)^2+y^2)^{3/2}\sqrt{(x^2+y^2)}} - \frac{Sy}{a^2\sqrt{(x^2+y^2)}}.$$

RECONSTRUCTION

3 **§26: along MR, normal to MC, by Say/(((a−x)²+y²)^(3/2) √(x²+y²)) − Sy/(a²√(x²+y²))** ✓
tMR = S·a·y / (dist³·ρ) − S·y / (a²·ρ) from 3.1

3.1 the same forces carried to the normal MR *construction* ∘
tMR = S·a·y / (dist³·ρ) − S·y / (a²·ρ) from constr. tMR, constr. fM, constr. fC, constr. dist, constr. x, constr. y, constr. M, constr. S

NOTES

MR is normal to MC: for a particle on the surface it is the horizontal direction in the plane through the luminary.

The page XML reads *Praetera*; the spelling is kept as transcribed.

§27 The expansion, and the force towards the centre to two terms

Opera omnia II 31, p. 37

EULER

27. Tametsi istae expressiones tantopere sint compositae, ut parum ex iis ad usum deduci posse videatur, tamen si consideremus distantiam Lunae a Terra, multo magis autem distantiam Solis, vehementer excedere quantitatem Terrae, ac propterea quantitates x et y; per approximationem satis commodas formulas ex iis derivare licebit. Cum enim sit proxime

$$\frac{1}{((a-x)^2+y^2)^{3/2}} = (a^2-2ax+x^2+y^2)^{-3/2}$$

$$= \frac{1}{a^3} + \frac{3(2ax-xx-yy)}{2a^5} + \frac{15(2ax-xx-yy)^2}{8a^7},$$

loco $\frac{1}{((a-x)^2+y^2)^{3/2}}$ satis tuto substitui poterit $\frac{1}{a^3} + \frac{3x}{a^4} + \frac{3(4xx-yy)}{2a^5}$. Ex his autem obtinebitur vis, qua particula M praeter gravitatem a vi Solis sive Lunae in S existentis ad centrum Terrae C in directione MC urgetur,

$$= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sx(3yy-2xx)}{2a^4\sqrt{(x^2+y^2)}}.$$

ENGLISH

27. Although these expressions are so complex that it seems little can be derived from them for practical use, nevertheless, if we consider that the distance of the Moon from the Earth, and much more so the distance of the Sun, greatly exceeds the size of the Earth, and therefore the quantities x and y, it will be possible to derive sufficiently convenient formulas from them by approximation. For since, approximately,

$$\frac{1}{((a-x)^2+y^2)^{3/2}} = (a^2-2ax+x^2+y^2)^{-3/2}$$

$$= \frac{1}{a^3} + \frac{3(2ax-xx-yy)}{2a^5} + \frac{15(2ax-xx-yy)^2}{8a^7},$$

in place of $\frac{1}{((a-x)^2+y^2)^{3/2}}$ one can quite safely substitute $\frac{1}{a^3} + \frac{3x}{a^4} + \frac{3(4xx-yy)}{2a^5}$. From these, moreover, is obtained the force by which a particle M, in addition to gravity, is urged toward the center of the Earth C in the direction MC by the force of the Sun or Moon located at S,

$$= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sx(3yy-2xx)}{2a^4\sqrt{(x^2+y^2)}}.$$

RECONSTRUCTION

4 **§27: with 1/((a−x)²+y²)^(3/2) put as 1/a³ + 3x/a⁴ + 3(4xx − yy)/2a⁵** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) + 3·S·x·(3·y² − 2·x²) / (2·a⁴·ρ) (to second order in ρ / a) from 4.1

4.1 tMC ≈ S·(y² − 2·x²) / (a³·ρ) + 3·S·x·(3·y² − 2·x²) / (2·a⁴·ρ) (to second order in ρ / a) *expansion* ≈
from 2, 1

NOTES

The small quantity is the size of the earth over the distance of the luminary. Collected by powers of a, Euler's series gives the term in a^{-5} as $\frac{3(-xx-yy)}{2a^5} + \frac{15\cdot 4xx}{8a^5} = \frac{3(4xx-yy)}{2a^5}$: the substitution he states.

The reconstruction does not redo the expansion symbolically. It tests that Euler's two-term expression and the exact one differ by a relative amount that falls like the square of the small ratio.

§27 The force across the vertical to two terms, and the editor's correction

Opera omnia II 31, p. 37

EULER

Praeterea autem eadem particula M sollicitabitur in directione MR ad MC normali, vi[5])

$$= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sy(4xx+yy)}{2a^4\sqrt{(x^2+y^2)}} = \frac{3Sy}{a^3\sqrt{(x^2+y^2)}}\left(x + \frac{4xx+yy}{2a}\right).$$

5) Editio princeps et Manuscriptus: $\frac{4xx-yy}{2a}$ loco $\frac{4xx+yy}{2a}$. Correxit EJA.

ENGLISH

Furthermore, this same particle M will be urged in the direction MR perpendicular to MC by the force[5])

$$= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sy(4xx+yy)}{2a^4\sqrt{(x^2+y^2)}} = \frac{3Sy}{a^3\sqrt{(x^2+y^2)}}\left(x + \frac{4xx+yy}{2a}\right).$$

5) First edition and Manuscript: $\frac{4xx-yy}{2a}$ instead of $\frac{4xx+yy}{2a}$ Corrected by EJA.

RECONSTRUCTION

5 **§27: the same for MR, as the first edition and the manuscript read: 3Sy/(a³√(x²+y²)) · (x + (4xx − yy)/2a)** ✓
tMR ≈ 3·S·x·y / (a³·ρ) + 3·S·y·(4·x² − y²) / (2·a⁴·ρ) (to second order in ρ / a) from 5.1

5.1 tMR ≈ 3·S·x·y / (a³·ρ) + 3·S·y·(4·x² − y²) / (2·a⁴·ρ) (to second order in ρ / a) *expansion* ≈
from 3, 1

NOTES

The printed text and the reconstruction differ here. Opera omnia II 31 prints $4xx + yy$; its note 5 says that the first edition and the manuscript have $4xx - yy$ and that the editor corrected it. The reconstruction follows the first edition and the manuscript, which are known here only through that note.

Reason: multiply the exact expression of §26, $\frac{Sy}{\sqrt{x^2+y^2}}\left(\frac{a}{d^3} - \frac{1}{a^2}\right)$, by Euler's own substitution for $1/d^3$ three lines above. The second term comes out with $4xx - yy$.

Test against the exact expression, the small ratio at 0.01, 0.001, 0.0001: with $4xx - yy$ the relative difference is 1.8e-4, 1.8e-6, 1.8e-8 (second order); with $4xx + yy$ it is 2.0e-2, 2.0e-3, 2.0e-4 (first order only). The emended formula is the argument `euler-tide-force-emendation`, which fails the check as it should.

§27 Both forces to first order

Opera omnia II 31, p. 37

EULER

Atque cum in his formulis termini primi posteriores multis vicibus excedant, rem crassius inspiciendo, particula M a vi Solis Lunaeve secundum MC urgebitur vi $= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}}$, in directione vero MR vi $= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}}$.

ENGLISH

And since in these formulas the first terms exceed the subsequent ones many times over, to a rougher approximation, the particle M will be urged by the force of the Sun or Moon along MC with a force $= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}}$, and in the direction MR with a force $= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}}$.

RECONSTRUCTION

6 **§27: along MC the particle is urged by S(yy − 2xx)/(a³√(x²+y²))** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 6.1

6.1 the first terms exceed the later ones many times *expansion* ≈
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 4

7 **§27: along MR by 3Sxy/(a³√(x²+y²))** ✓
tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) from 7.1

7.1 tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) *expansion* ≈
from 5

NOTES

Rem crassius inspiciendo: only the first term of each formula is kept. These two expressions are the conclusion of the argument; §§28–29 read consequences off them.

The question of the emendation does not touch this result: both readings have the same first term.

§28 Where gravity is increased and where diminished

Opera omnia II 31, p. 37–38

EULER

28. Ex his igitur postremis formulis intelligitur ab actione Solis sive Lunae in S existentis gravitatem particulae M augeri, si eius situs respectu rectae SC ita fuerit comparatus, ut sit $yy > 2xx$ hoc est tangens anguli $MCP > \sqrt{2}$ posito sinu toto $= 1$, contra vero gravitatem diminui, si fuerit $yy < 2xx$. Quare cum /38/ angulus cuius tangens est $= \sqrt{2}$ contineat 54° 45′ circiter, si concipiatur circulus Terrae maximus quicunque $ADBE$ [fig. 4], cuius planum per punctum S transeat, in eoque ducantur rectae FCI et GCH, quae cum recta SAB angulos constituant 54° 45′; tum omnes Terrae particulae in spatiis FCH et GCI sitae gravitatis naturalis

augmentum accipient, reliquae vero particulae in spatiis FCG et HCI positae decrementum gravitatis patientur. Atque hinc, quacumque Terrae particula proposita, definiri poterit, quantum eius gravitas a Sole Lunave in S existente vel augeatur vel diminuatur.

ENGLISH

28. From these latter formulas, therefore, it is understood that by the action of the Sun or Moon located at S, the gravity of the particle M is increased if its position with respect to the straight line SC is such that $yy > 2xx$, that is, the tangent of the angle $MCP > \sqrt{2}$ setting the total sine $= 1$; but on the contrary, the gravity is diminished if $yy < 2xx$. Wherefore, since *|38|* the angle whose tangent is $= \sqrt{2}$ contains approximately $54^\circ\ 45'$, if we conceive any great circle of the Earth $ADBE$ [fig. 4], whose plane passes through the point S, and in it are drawn the straight lines FCI and GCH, which make angles of $54^\circ\ 45'$ with the straight line SAB; then all the particles of the Earth situated in the spaces FCH and GCI will receive an increase in natural gravity, while the remaining particles placed in the spaces FCG and HCI will suffer a decrease in gravity. And from this, for any given particle of the Earth, it can be defined how much its gravity is either increased or diminished by the Sun or Moon existing at S.

RECONSTRUCTION

8 **§§28–29: gravity is increased where yy > 2xx, most at D and E, by Sρ/a³; decreased where yy < 2xx, most at A and B, by 2Sρ/a³** ✓
S·(y² − 2·x²) / (a³·ρ) = S·ρ / (a³)·(1 − 3·cos(φ)·cos(φ)) from 8.3

- **8.1** x = ρ·cos(φ) — *construction* ∘
  from constr. x
- **8.2** y = ρ·sin(φ) — *construction* ∘
  from constr. y
- **8.3** S·(y² − 2·x²) / (a³·ρ) = S·ρ / (a³)·(1 − 3·cos(φ)·cos(φ)) — *algebra* ≈
  from 8.1, 8.2

NOTES

With $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ the force towards the centre is $\frac{S\rho}{a^3}(1 - 3\cos^2\varphi)$; it changes sign where $\cos^2\varphi = \frac{1}{3}$, that is $\tan\varphi = \sqrt{2}$.

The angle whose tangent is $\sqrt{2}$ is 54°44′ to the nearest minute (54.7356°); Euler gives 54°45′ *circiter*.

Euler's Fig. 4, with the lines FCI and GCH, is not drawn in the reconstruction.

§28 The direction of the horizontal force

Opera omnia II 31, p. 38

EULER

Altera vero vis, qua particula M [fig. 3] in directione horizontali MR urgetur, affirmativa erit, in eamque plagam, quae in figura repraesentatur, verget, si quantitates x et y ambae fuerint vel affirmativae vel negativae: contrariumque eveniet, si earum altera sit affirmativa, altera negativa. Quare si particula M [fig. 4] sita fuerit vel in quadrante ACD vel ACE, tum vis horizontalis ad rectam CA tendet; contra vero haec vis ad radium CB dirigetur, si particula M sit vel in quadrante BCD vel BCE constituta. Ex quibus perspicitur effectus vel Solis vel Lunae in ambo hemisphaeria, superius scilicet DAE et inferius DBE, inter se esse fere similes; quae similitudo quoque in ipso aestu Maris observatur.

ENGLISH

But the other force, by which the particle M [fig. 3] is urged in the horizontal direction MR, will be positive, and will tend toward that region which is represented in the figure, if the quantities x and y are both either positive or negative; and the opposite will happen if one of them is positive and the other negative. Therefore, if the particle M [fig. 4] is situated either in the quadrant ACD or ACE, then the horizontal force will tend toward the straight line CA; on the contrary, this force will be directed toward the radius CB if the particle M is located either in the quadrant BCD or BCE. From which it is perceived that the effects of either the Sun or the Moon on both hemispheres, namely the upper DAE and the lower DBE, are almost similar to each other; which similarity is also observed in the tide of the Sea itself.

RECONSTRUCTION

Read off the horizontal force: 3Sxy/a³ has the sign of xy, that is of sin 2φ. No step of its own.

Read off step 9.

NOTES

This is the passage that connects the geometry with the phenomenon: the force is alike in the hemisphere under the luminary and in the opposite one, and so are the two tides of a day.

§29 On the surface: the greatest values

Opera omnia II 31, p. 38–39

EULER

29. Ponamus nunc particulam M in ipsa Terrae superficie esse constitutam, eritque $\sqrt{(x^2 + y^2)} = 1$ ob Terrae semidiametrum $= 1$. Quare si particula M *|39|* fuerit posita in M, existente anguli ACM sinu $= y$ et cosinu $= x$, eius gravitas naturalis acceleratrix a Sole Lunave in S augebitur vi $= \frac{S(yy-2xx)}{a^3}$, secundum horizontem autem in directione MR urgebitur vi $= \frac{3Sxy}{a^3}$. Gravitas igitur maxime augebitur, si particula M posita fuerit in D vel E, quibus in locis punctum S in horizonte apparet; ibi vero gravitatis augmentum erit $= \frac{S}{a^3}$. In punctis autem A et B, quae punctum S vel in suo zenith vel nadir positum habent, maximum deprehendetur gravitatis decrementum, quod scilicet erit $= \frac{2S}{a^3}$: ita ut maximum gravitatis decrementum duplo maius sit quam maximum incrementum. Vis autem horizontalis $\frac{3Sxy}{a^3}$ maxima evadet, si angulus ACM fuerit semirectus, id quod accidit in iis Terrae regionibus, in quibus punctum S conspicitur vel 45° gradibus supra horizontem elevatum, vel tantundem sub horizonte depressum latet: his igitur casibus ob $xy = \frac{1}{2}$

fiet vis horizontalis $= \frac{3S}{2a^3}$. Huius ergo vis effectus in hoc consistet, ut directio gravitatis mutetur, atque versus rectam SC inclinetur angulo cuius tangens est $= \frac{3S}{2a^3}$, existente sinu toto $= 1$, quia gravitatem unitate designamus.

ENGLISH

29. Let us now assume that the particle M is situated on the very surface of the Earth, and we will have $\sqrt{(x^2 + y^2)} = 1$ because of the Earth's semidiameter $= 1$. Therefore, if the particle M |39| is placed at M, with the sine of the angle ACM being $= y$ and the cosine $= x$, its natural accelerating gravity from the Sun or Moon at S will be increased by the force $= \frac{S(yy-2xx)}{a^3}$, while along the horizon in the direction MR it will be urged by the force $= \frac{3Sxy}{a^3}$. Therefore, gravity will be increased the most if the particle M is placed at D or E, in which places the point S appears on the horizon; there, indeed, the increase of gravity will be $= \frac{S}{a^3}$. But at points A and B, which have the point S placed either in their zenith or nadir, the greatest decrease of gravity will be detected, which will namely be $= \frac{2S}{a^3}$: so that the maximum decrease of gravity is twice as great as the maximum increase. Moreover, the horizontal force $\frac{3Sxy}{a^3}$ will become greatest if the angle ACM is a semi-right angle, which happens in those regions of the Earth in which the point S is seen either elevated 45 degrees above the horizon, or lies hidden depressed by the same amount below the horizon: in these cases, therefore, because of $xy = \frac{1}{2}$, the horizontal force will become $= \frac{3S}{2a^3}$. The effect of this force, therefore, will consist in this, that the direction of gravity is changed, and is inclined towards the straight line SC by an angle whose tangent is $= \frac{3S}{2a^3}$, the total sine being $=$ 1, since we designate gravity by unity.

RECONSTRUCTION

9 **§29: the horizontal force is greatest where the angle ACM is half a right angle** ✓
3·S·x·y / (a³·ρ) = 3·S·ρ / (2·a³)·sin(2·φ) from 9.3

- **9.1** x = ρ·cos(φ) — *construction* ∘
  from constr. x
- **9.2** y = ρ·sin(φ) — *construction* ∘
  from constr. y
- **9.3** 3·S·x·y / (a³·ρ) = 3·S·ρ / (2·a³)·sin(2·φ) — *algebra* ≈
  from 9.1, 9.2

Read off step 8.

NOTES

With the semidiameter of the earth as unit: at D and E ($\varphi = 90°$) gravity is increased by S/a^3; at A and B ($\varphi = 0°, 180°$) it is diminished by $2S/a^3$; the horizontal force is greatest at $\varphi = 45°$, where it is $3S/2a^3$. All three values follow from the two expressions of the steps cited.

The checks report $\varphi = 0°$ and 180° as states the construction does not cover: there M lies on CS and the perpendicular MP has no direction. Euler's value at A and B is the value of the formula there.

THEREFORE

∴ **the vertical component of the tide-raising force** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 6

∴ **and its horizontal component** ≈
tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) from 7

4 What the reconstruction shows

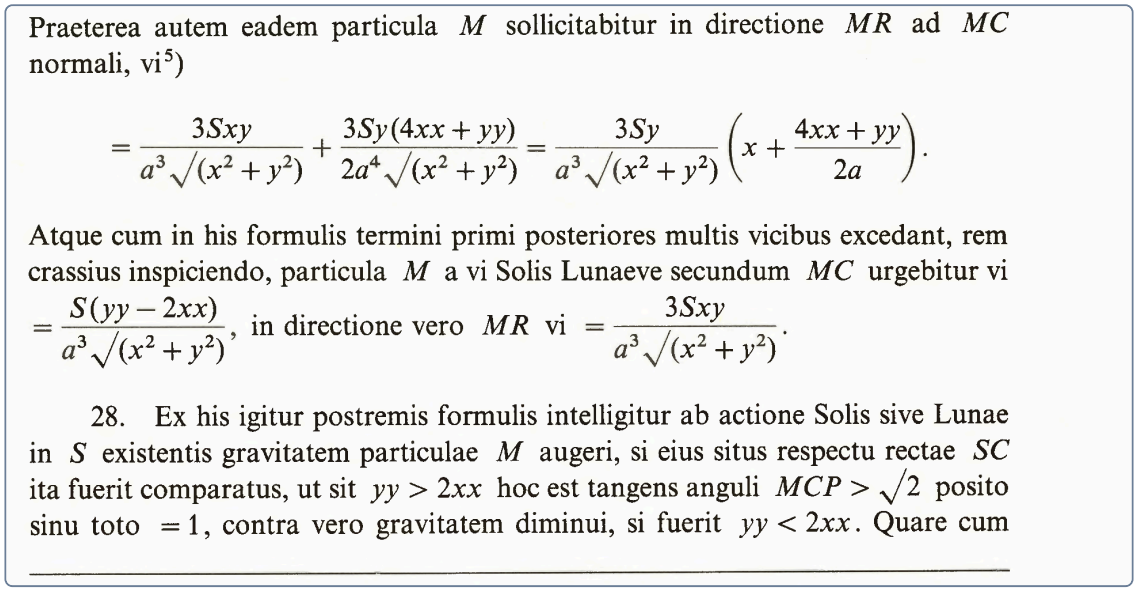
The principle is a definition. The sentence at the end of §22 is not a premise among others. It fixes what the argument is about: the force on the particle less the force on the centre. In the model it is the definition of the two quantities observed, along MC and along MR, and every later step is a statement about them.

The exact components are confirmed, by another route than Euler's. §26 reaches the two components through a perpendicular QV and the similar triangles QVM and MPC. The reconstruction does not rebuild this. It states that Euler's closed expressions equal the quantities defined, and the check confirms it in every sampled configuration. This is the widest gap between the source and the reconstruction, and the notes say so at the passage.

The expansion has an order, and the order can be tested. §27 keeps first two terms and then one. A step of this kind is not an equality. It is checked by making the ratio of the particle's distance to the luminary's distance small, a hundredth, a thousandth, a ten-thousandth, and requiring the relative difference to fall like the stated power of the ratio.

An emendation of the editor is wrong. For the component along MR the Opera omnia print 4xx + yy in the term of second order, with the note "Editio princeps et Manuscriptus: (4xx − yy)/2a loco (4xx + yy)/2a. Correxit EJA." (Figure 1). The reconstruction follows the first edition and the manuscript. Their reading follows from Euler's own substitution three lines above, and it is of second order: its relative difference from the exact component is $1.8 \cdot 10^{-4}$, $1.8 \cdot 10^{-6}$ and $1.8 \cdot 10^{-8}$ at the three ratios. The emended reading is of first order only:

$2.0 \cdot 10^{-2}$, $2.0 \cdot 10^{-3}$ and $2.0 \cdot 10^{-4}$. The emendation does not touch Euler's results, which use the first term. We have not seen the first edition or the manuscript; their reading is known to us from the editor's note.



Praeterea autem eadem particula M sollicitabitur in directione MR ad MC normali, vi[5])

$$= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sy(4xx+yy)}{2a^4\sqrt{(x^2+y^2)}} = \frac{3Sy}{a^3\sqrt{(x^2+y^2)}}\left(x + \frac{4xx+yy}{2a}\right).$$

Atque cum in his formulis termini primi posteriores multis vicibus excedant, rem crassius inspiciendo, particula M a vi Solis Lunaeve secundum MC urgebitur vi $= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}}$, in directione vero MR vi $= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}}$.

28. Ex his igitur postremis formulis intelligitur ab actione Solis sive Lunae in S existentis gravitatem particulae M augeri, si eius situs respectu rectae SC ita fuerit comparatus, ut sit $yy > 2xx$ hoc est tangens anguli $MCP > \sqrt{2}$ posito sinu toto $= 1$, contra vero gravitatem diminui, si fuerit $yy < 2xx$. Quare cum

Figure 1 Opera omnia II 31, p. 37 (`fr:enestroem/E57/0018`): the component along MR with the emended sign, the first-order formulae, and note 5.

A rounded angle. §28 gives the angle whose tangent is $\sqrt{2}$ as 54°45′ "circiter". To the nearest minute it is 54°44′.

What is not reconstructed. The first resolution of §25, along CS and along MP with the diagonal MO, is left out: §26 replaces it, and no formula of §§26 to 29 cites it. Euler's Fig. 4 is not drawn. The checks report the positions of the particle on the line CS as not covered by the construction of the perpendicular MP.

5 The exposition as a format

The page is built from one file that names, for every passage, where it begins and where it ends on a page of the edition, which steps reconstruct it, and the notes. Nothing of the source is typed into that file. The builder takes the Latin from the page of the edition and the English from the stored translation, and it refuses to build when a passage is not found exactly once, when a step of the argument stands beside no passage, or when a passage lies in no paragraph of the reference register. A correction of the translation is listed with its reason and shown as a correction. The reference of the source comes from a checked record.

The same file gives the interactive page and the print, so the two cannot drift apart. The procedure for another source is the same: choose the segment, write the model, write the argument, divide the text into passages, attach the steps, and annotate what remains.

6 Limits

The Latin is the edition's transcription, not a new collation; one spelling of the transcription (*Praetera*) is kept as it stands. The English is a machine translation compared with the Latin by the author, not a scholarly translation. The reconstruction covers what Euler argues in these sections, not what he presupposes from the sections before: that the earth may be treated as if collected in its centre (the paragraph before §22), and the form of the force. The emendation is judged on the argument alone, without the first edition or the manuscript.

Availability

The exposition is `expositions/euler-e57-tide.yaml` in the Famulus workspace `famulus-astronomica`; the model, the derivation, the tested emendation and the figure are in `examples/euler/`. The page element

is `<astra-exposition>`. Euler is cited by the pages of the Famulus collection `enestroem`, document `E57`; page n of the collection is page n + 19 of Opera omnia II 31.

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[2] G. Graßhoff, "Two Geometrical Arguments Reconstructed. Euler on the Tide-Raising Force, and the Equivalence of Eccentric and Epicycle," 2026.

[3] G. Graßhoff, "Geometrical Reasoning in the History of Astronomy. Fundamentals," 2026.

[4] G. Graßhoff, "astra/1. A Language for Historical Astronomical Models and the Geometrical Reasoning about Them," 2026.