Two Geometrical Arguments Reconstructed

Euler on the Tide-Raising Force, and the Equivalence of Eccentric and Epicycle in Swerdlow's Exposition of Ptolemy

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Abstract

Two arguments from the history of astronomy are reconstructed with the same method: the model is made explicit, the argument is written as steps with premises and rules, and a program checks every step. The first, treated briefly, is Euler's derivation of the force by which the sun or the moon tends to raise the tides, in his prize essay of 1740 (Eneström 57). Euler's exact components and his expansion in the small ratio of the earth's size to the distance of the luminary are confirmed. In the term of second order the edition in the Opera omnia prints 4xx + yy, an emendation of the editor against the first edition and the manuscript, which have 4xx − yy. The check decides between them: Euler's original reading is of second order, the emended one of first order only. The second argument, treated in full, is the equivalence of the eccentric and the epicyclic form of Ptolemy's solar model as Noel Swerdlow sets it out in *The Renaissance of Astronomy* (2026). The explicit geometry of both forms is given first, with the astronomical meaning of every object. The argument is then displayed with its sub-arguments and cases: off the apsidal line the two suns coincide by a parallelogram (Elements I.28 and I.33); at the apogee and at the perigee, where the parallelogram collapses, by adding and subtracting lengths along one line. Both hypotheses on the radii are needed, and none is used tacitly.

Keywords history of astronomy · Euler · tides · Ptolemys solar model · geometrical reasoning · argument reconstruction

1 Two cases, one method

A geometrical argument in an astronomical text has a figure, a few sentences, and inferences that are marked by "therefore" and little else. To reconstruct it is to say what is argued, from what, and by what right. The method used here has three parts [1]. The *model* is written out: every point, line and circle with its construction and its meaning, lengths and angles as symbols without values. The *argument* is written as steps: one proposition each, with the premises it is drawn from and the rule that licenses it, grouped into sub-arguments. And a program *checks* the result: whether every premise is there, whether every proposition is true in the model for every admissible value and every position, whether every step is an instance of the rule it names, and whether the conclusion rests only on what is cited. The language is astra/1 [2].

The two cases are far apart. Euler's argument is from 1740, about forces, and ends in an approximation. The second is ancient in substance, about two kinematic models, and exact. That the same method serves both is the point of putting them side by side. The first is kept short and serves as a demonstration. The second is given in full.

2 Euler: the force that raises the tides

2.1 The passage

In chapter II of the *Inquisitio physica in causam fluxus ac refluxus maris* [3] Euler asks by what force the sun or the moon tends to move a particle of the earth relative to the earth's centre. The principle is stated in §22: the force proper to a particle is found "si a vi acceleratrice illam particulam sollicitante auferatur vis acceleratrix, quam centrum Terrae patitur" — if from the accelerating force on the particle the accelerating force on the centre is taken away. §§24 to 27 carry this out with one figure, Fig. 3 (`fr:enestroem/E57/0015/fig01`).

2.2 The model

Euler's figure is a model in the sense used here (Table 1, Figure 1): the centre C, the luminary S at the distance a, a particle M placed by CP = x and PM = y, and the forces as quantities defined on these. One angle varies: the angle ACM between the line to the luminary and the line to the particle.

Table 1 Euler's geometry of the tide-raising force, explicit (`fr:astra/euler-tide-force`). The last two observables are the quantities the argument is about: the force on the particle less the force on the centre, taken along MC and along MR.

	what it is	defined by	astronomical meaning
a	parameter, length	in (30, 1000)	distance CS of the luminary from the centre of the earth
ρ	parameter, length	in (0.5, 2)	distance CM of the particle from the centre of the earth
S	parameter, quantity	any value	absolute force of the luminary: its accelerating force at the distance of one semidiameter of the earth
φ	kinematic variable	free	the angle ACM between the line to the luminary and the line to the particle
C	point	`observer()`	centre of the earth
S	point	`C + polar(a, 90°)`	the sun or the moon
A	point	`C + polar(rho, 90°)`	the point of the sphere through M that has the luminary in its zenith
B	point	`C - polar(rho, 90°)`	the point that has it in its nadir
D	point	`C + polar(rho, 180°)`	a point that has it in the horizon
E	point	`C + polar(rho, 0°)`	the opposite point that has it in the horizon
shell	circle	`circle(C, rho)`	the sphere about C through the particle
M	point	`C + polar(rho, 90° + phi)`	a particle of the earth
P	point	`foot(M, line(C, S))`	foot of the perpendicular from M on CS
CS	observable	`direction(C, S)`	direction of the luminary from the centre
x	observable	`rho * cos(phi)`	CP, counted towards the luminary
y	observable	`rho * sin(phi)`	PM, counted towards D
dist	observable	`distance(M, S)`	SM, the distance of the particle from the luminary
fM	observable	`Sf / (dist * dist)`	accelerating force on the particle towards S
fC	observable	`Sf / (a * a)`	accelerating force on the centre towards S

	what it is	defined by	astronomical meaning
tMC	observable	`fM * cos(direction(M, S) - direction(M, C)) - fC * cos(direction(C, S) - direction(M, C))`	the force on the particle less the force on the centre, along MC, towards the centre
tMR	observable	`fM * cos(direction(M, S) - (direction(M, C) + 90°)) - fC * cos(direction(C, S) - (direction(M, C) + 90°))`	the same along MR, normal to MC, towards the side of the luminary

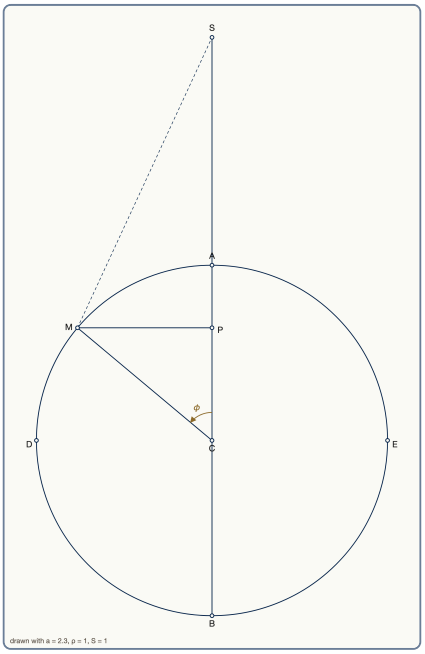


Figure 1 After Euler's Fig. 3: the sphere about C through the particle M, the luminary S, the perpendicular MP on CS. The dotted line MS closes the right triangle MPS of §24. Euler's auxiliary points Q, O, R, V, N are not drawn. In the printed figure S stands at about two radii; so it does here, as a drawing value.

2.3 The argument

Figure 2 gives Euler's argument in nine parts, each a paragraph or a sentence of his.

The tide-raising force along and across the vertical, to first order in the earth's size over the luminary's distance

ARGUMENT

1 **§24: $SM^2 = (a - x)^2 + y^2$** ✓
$dist^2 = (a - x)\cdot(a - x) + y^2$ from 1.6

- 1.1 MP is normal to CS — *construction* ∘
  $MP \perp SC$ from constr. P
- 1.2 the right triangle MPS — *Elements I.47* ✓
  $MS\cdot MS = MP\cdot MP + SP\cdot SP$ from 1.1
- 1.3 $SP = a - x$ — *construction* ∘
  from constr. S, constr. P, constr. x
- 1.4 $PM = y$ — *construction* ∘
  $MP = |y|$ from constr. M, constr. P, constr. y
- 1.5 dist = MS — *construction* ∘
  from constr. dist
- 1.6 $dist^2 = (a - x)\cdot(a - x) + y^2$ — *algebra* ≈
  from 1.2, 1.3, 1.4, 1.5

2 **§26: towards the centre the particle is urged by $Sx/(a^2\sqrt{(x^2+y^2)}) - S(ax - xx - yy)/(((a-x)^2+y^2)\hat{}(3/2))\sqrt{(x^2+y^2)}$** ✓
$tMC = S\cdot x / (a^2\cdot\rho) - S\cdot(a\cdot x - x^2 - y^2) / (dist^3\cdot\rho)$ from 2.1

2.1 the force towards S resolved along CS and along MC, the force on the centre taken away, and what remains along MQ carried to MC by the similar triangles QVM and MPC — *construction* ∘
tMC = S·x / (a²·ρ) − S·(a·x − x² − y²) / (dist³·ρ) from constr. tMC, constr. fM, constr. fC, constr. dist, constr. x, constr. y, constr. M, constr. S

3 **§26: along MR, normal to MC, by Say/(((a−x)²+y²)^(3/2) √(x²+y²)) − Sy/(a²√(x²+y²))** ✓
tMR = S·a·y / (dist³·ρ) − S·y / (a²·ρ) from 3.1

3.1 the same forces carried to the normal MR — *construction* ∘
tMR = S·a·y / (dist³·ρ) − S·y / (a²·ρ) from constr. tMR, constr. fM, constr. fC, constr. dist, constr. x, constr. y, constr. M, constr. S

4 **§27: with 1/((a−x)²+y²)^(3/2) put as 1/a³ + 3x/a⁴ + 3(4xx − yy)/2a⁵** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) + 3·S·x·(3·y² − 2·x²) / (2·a⁴·ρ) (to second order in ρ / a) from 4.1

4.1 tMC ≈ S·(y² − 2·x²) / (a³·ρ) + 3·S·x·(3·y² − 2·x²) / (2·a⁴·ρ) (to second order in ρ / a) — *expansion* ≈
from 2, 1

5 **§27: the same for MR, as the first edition and the manuscript read: 3Sy/(a³√(x²+y²)) · (x + (4xx − yy)/2a)** ✓
tMR ≈ 3·S·x·y / (a³·ρ) + 3·S·y·(4·x² − y²) / (2·a⁴·ρ) (to second order in ρ / a) from 5.1

5.1 tMR ≈ 3·S·x·y / (a³·ρ) + 3·S·y·(4·x² − y²) / (2·a⁴·ρ) (to second order in ρ / a) — *expansion* ≈
from 3, 1

6 **§27: along MC the particle is urged by S(yy − 2xx)/(a³√(x²+y²))** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 6.1

6.1 the first terms exceed the later ones many times — *expansion* ≈
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 4

7 **§27: along MR by 3Sxy/(a³√(x²+y²))** ✓
tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) from 7.1

7.1 tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) — *expansion* ≈
from 5

8 **§§28–29: gravity is increased where yy > 2xx, most at D and E, by Sρ/a³; decreased where yy < 2xx, most at A and B, by 2Sρ/a³** ✓
S·(y² − 2·x²) / (a³·ρ) = S·ρ / (a³)·(1 − 3·cos(φ)·cos(φ)) from 8.3

8.1 x = ρ·cos(φ) — *construction* ∘
from constr. x

8.2 y = ρ·sin(φ) — *construction* ∘
from constr. y

8.3 S·(y² − 2·x²) / (a³·ρ) = S·ρ / (a³)·(1 − 3·cos(φ)·cos(φ)) — *algebra* ≈
from 8.1, 8.2

9 **§29: the horizontal force is greatest where the angle ACM is half a right angle** ✓
3·S·x·y / (a³·ρ) = 3·S·ρ / (2·a³)·sin(2·φ) from 9.3

9.1 x = ρ·cos(φ) — *construction* ∘
from constr. x

9.2 y = ρ·sin(φ) — *construction* ∘
from constr. y

9.3 3·S·x·y / (a³·ρ) = 3·S·ρ / (2·a³)·sin(2·φ) — *algebra* ≈
from 9.1, 9.2

THEREFORE

∴ **the vertical component of the tide-raising force** ✓
tMC ≈ S·(y² − 2·x²) / (a³·ρ) (to first order in ρ / a) from 6

∴ **and its horizontal component** ≈
tMR ≈ 3·S·x·y / (a³·ρ) (to first order in ρ / a) from 7

Figure 2 Euler's argument for the tide-raising force (E57 §§24–29), reconstructed. ✓ an instance of the rule named, ∘ read off the construction, ≈ by calculation or expansion.

Three things are checked here that a reader usually takes on trust.

The exact components. §26 gives the two components before any approximation: towards the centre Sx/(a²√(x²+y²)) − S(ax − xx − yy)/(((a−x)²+y²)^(3/2)√(x²+y²)), and along MR Say/(((a−x)²+y²)^(3/2)√(x²+y²)) − Sy/(a²√(x²+y²)). The model defines the same two quantities geometrically, as the difference of two forces projected on two directions. Parts 2 and 3 state that Euler's expressions equal them. They do, in every sampled configuration.

The expansion. §27 replaces 1/((a−x)²+y²)^(3/2) by 1/a³ + 3x/a⁴ + 3(4xx − yy)/2a⁵ and obtains each component as a term in 1/a³ and a term in 1/a⁴, then keeps the first. A step of this kind is not an equality. It is written as `approximately(…, small: ρ / a, order: n)` and checked by making the ratio of the particle's distance to the luminary's distance small — a hundredth, a thousandth, a ten-thousandth — and requiring the relative difference to fall like the n-th power of the ratio. Parts 4 and 5 are of second order, parts 6 and 7 of first order.

The consequences. §§28 and 29 read off that gravity is increased where yy > 2xx and decreased where yy < 2xx, most at the points that have the luminary in the horizon and in the zenith, and that the horizontal force is greatest "si angulus ACM fuerit semirectus". Parts 8 and 9 put the two first-order components in the forms $(S\rho/a^3)(1 - 3\cos^2\varphi)$ and $(3S\rho/2a^3)\sin 2\varphi$, from which all of this follows.

2.4 An emendation, tested

In part 5 the reconstruction follows the first edition and the manuscript, not the edition it is read in. The Opera omnia print the component along MR as

$$3Sxy/(a^3\sqrt{(x^2+y^2)}) + 3Sy(4xx + yy)/(2a^4\sqrt{(x^2+y^2)}),$$

with the note: "Editio princeps et Manuscriptus: (4xx − yy)/2a loco (4xx + yy)/2a. Correxit EJA." (Figure 3; the volume's editors are E. J. Aiton and A. Kleinert). The editor has changed a sign.

Praeterea autem eadem particula M sollicitabitur in directione MR ad MC normali, vi[5])

$$= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}} + \frac{3Sy(4xx+yy)}{2a^4\sqrt{(x^2+y^2)}} = \frac{3Sy}{a^3\sqrt{(x^2+y^2)}}\left(x + \frac{4xx+yy}{2a}\right).$$

Atque cum in his formulis termini primi posteriores multis vicibus excedant, rem crassius inspiciendo, particula M a vi Solis Lunaeve secundum MC urgebitur vi $= \frac{S(yy-2xx)}{a^3\sqrt{(x^2+y^2)}}$, in directione vero MR vi $= \frac{3Sxy}{a^3\sqrt{(x^2+y^2)}}$.

28. Ex his igitur postremis formulis intelligitur ab actione Solis sive Lunae in S existentis gravitatem particulae M augeri, si eius situs respectu rectae SC ita fuerit comparatus, ut sit $yy > 2xx$ hoc est tangens anguli $MCP > \sqrt{2}$ posito sinu toto $= 1$, contra vero gravitatem diminui, si fuerit $yy < 2xx$. Quare cum

Figure 3 Opera omnia II 31, p. 37 (`fr:enestroem/E57/0018`): the component along MR with the emended sign, the first-order formulae, and note 5.

The question which reading is right can be put to the model. Euler's exact component along MR stands one page earlier, and three lines above the formula the same page gives the substitution with 4xx − yy. Stated as a proposition of second order, the reading of the first edition and the manuscript holds: its relative difference from the exact component is $1.8 \cdot 10^{-4}$, $1.8 \cdot 10^{-6}$ and $1.8 \cdot 10^{-8}$ at the three ratios, falling by a hundred for every ten. The emended reading does not (Listing 1): its relative difference is $2.0 \cdot 10^{-2}$, $2.0 \cdot 10^{-3}$ and $2.0 \cdot 10^{-4}$, falling by ten only. It agrees with the exact component to first order and no better; its second term makes the formula worse than the first term alone. The same follows by hand: multiplying Euler's substitution by a and taking $1/a^2$ away leaves $3x/a^3 + 3(4xx - yy)/2a^4$.

Listing 1 The emended formula stated as a proposition of second order (`fr:astra/euler-tide-force-emendation`). The text is written to fail, and fails at the level *true*.

```
$ famulus-astra derive euler-tide-force-emendation
euler-tide-force-emendation: derivation in euler-tide-force, level grounded (2/5)
  ✗ true
      conclusion: approximately(tMR, 3 * Sf * x * y / (a * a * a * rho) + 3 * Sf *
            y * (4 * x * x + y * y) / (2 * a * a * a * a * rho)) fails (not of
            order 2: relative difference 2.0e-02 at ratio 0.01, 2.0e-03 at ratio
            0.001, 2.0e-04 at ratio 0.0001)
  ✗ conclusive
      the conclusion does not hold
```

```
✗ conclusion: approximately(tMR, 3 * Sf * x * y / (a * a * a * rho) + 3 * Sf * y
      * (4 * x * x + y * y) / (2 * a * a * a * a * rho)) from exact by expansion
      — false (not of order 2: relative difference 2.0e-02 at ratio 0.01,
      2.0e-03 at ratio 0.001, 2.0e-04 at ratio 0.0001)
```

Euler's sign was right, and the emendation introduces an error into the text. It does not touch his results, which use the first term only. We have not seen the first edition or the manuscript; their reading is known to us from the editor's note. Table 2 lists the sentences of Euler the reconstruction rests on.

Table 2 The sentences of Euler used here, with their paragraphs in the collection `enestroem`, document `E57`, and what each is checked as. ✕ marks the emended formula.

#	place	quoted	checked as	verdict
1	`0014/p03`	propria vis, qua quaeque particula agitabitur, innotescet, si a vi acceleratrice illam particul…	not formal –	—
2	`0016/p02`	SP = a - x et SM = ((a - x)^2 + y^2)	step `setting`	✓
3	`0017/p04`	unde omnino particula M a vi ad S tendente versus C urgebitur vi	step `exact_MC`	✓
4	`0017/p05`	eadem particula M in directione MR ad MC normali sollicitabitur vi	step `exact_MR`	✓
5	`0018/p04`	satis tuto substitui poterit 1/a^3 + 3x/a^4 + 3(4xx - yy)/2a^5	step `second_MC`	✓
6	`0018/p07`	(x + 4xx + yy/2a)	step `conclusion`	✕
7	`0018/p08`	particula M a vi Solis Lunaeve secundum MC urgebitur vi	step `first_MC`	✓
8	`0018/p08`	in directione vero MR vi	step `first_MR`	✓
9	`0018/p09`	ut sit yy > 2xx hoc est tangens anguli MCP >	step `vertical`	✓
10	`0020/p02`	Vis autem horizontalis 3Sxy/a^3 maxima evadet, si angulus ACM fuerit semirectus	step `horizontal`	✓

3 The equivalence of eccentric and epicycle

3.1 The passage

Swerdlow's account of Ptolemy's solar theory [4, p. 19] introduces the two forms of the model in one paragraph. "Ptolemy sets out eccentric and epicyclic models for this inequality", it says, and after describing the eccentric:

> "To construct the epicyclic model, complete parallelogram OESC and let C be the center of an epicycle of radius r = e moving uniformly through $\bar{\kappa}$ on a concentric circle about O of radius R′ = R while the sun S moves in the opposite direction through $\bar{\kappa}$ about C from OC extended to the apogee of the epicycle, so that r is always parallel to the apsidal line AB."

The next page draws the consequence: “the direction OS from the earth to the sun, the diagonal of the parallelogram, is the same in both models and thus the apparent nonuniform motion of the sun is identical.” The model is Ptolemy’s. The text whose use of it is analysed is Swerdlow’s. His Fig. 1.2 shows both forms in one diagram (Figure 4).

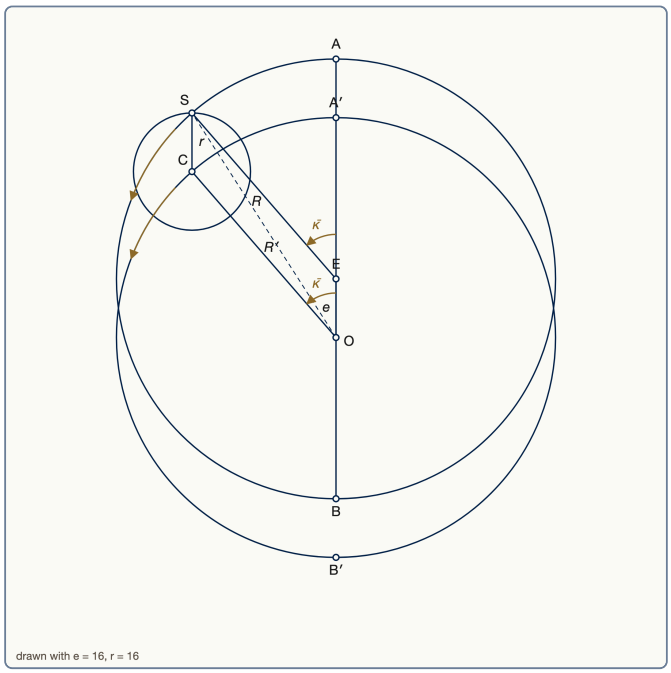


Figure 4 Swerdlow’s Fig. 1.2 redrawn from the texts: the eccentric form (centre E, sun S on the circle about E) and the epicyclic form (centre C of the epicycle on the circle about O) in one plane. The eccentricity is exaggerated, as in the printed figure.

3.2 The explicit geometry and its astronomical meaning

Before any argument, the two models (Table 3). Each is a pure geometry: it has parameter symbols and one kinematic variable, the mean anomaly, and no values.

Table 3 The two forms of Ptolemy’s solar model, explicit: `fr:astra/solar-eccentric` and `fr:astra/solar-epicyclic`.

Ecc — Ptolemy’s solar model, eccentric form (`fr:astra/solar-eccentric`)

	what it is	defined by	astronomical meaning
R	parameter, length	in (1, 1000)	radius of the eccentric
e	parameter, length	in (0, R / 2)	eccentricity, the distance OE
λ_A	parameter, angle	any value	longitude of the apogee
T	parameter, time	in (300, 400)	tropical year
$\bar{\kappa}_0$	parameter, angle	any value	mean anomaly at t = 0
$\bar{\kappa}$	kinematic variable	uniform in time, period T	mean anomaly about E, counted from the apogee
O	point	`observer()`	earth

	what it is	defined by	astronomical meaning
E	point	`O + polar(e, lamA)`	centre of the eccentric
ecc	circle	`circle(E, R)`	the eccentric
A	point	`E + polar(R, lamA)`	apogee
B	point	`E - polar(R, lamA)`	perigee
S	point	`E + polar(R, lamA + kappa_bar)`	sun
$\bar{\lambda}_S$	observable	`direction(E, S)`	mean longitude
λ_S	observable	`direction(O, S)`	true longitude
κ	observable	`lam - lamA`	true anomaly about O
c	observable	`lam_bar - lam`	equation of centre, mean minus true longitude

Epi — Ptolemy's solar model, epicyclic form (`fr:astra/solar-epicyclic`)

	what it is	defined by	astronomical meaning
R′	parameter, length	in (1, 1000)	radius of the deferent, the circle about O
r	parameter, length	in (0, R' / 2)	radius of the epicycle
λ_A	parameter, angle	any value	longitude of the apogee
T	parameter, time	in (300, 400)	tropical year
$\bar{\kappa}_0$	parameter, angle	any value	mean anomaly at t = 0
$\bar{\kappa}$	kinematic variable	uniform in time, period T	mean anomaly: the motion of C about O, and of the sun about C in the opposite sense
O	point	`observer()`	earth
def	circle	`circle(O, R')`	the deferent
A′	point	`O + polar(R', lamA)`	apogee of the deferent
B′	point	`O - polar(R', lamA)`	perigee of the deferent
C	point	`O + polar(R', lamA + kappa_bar)`	centre of the epicycle
epi	circle	`circle(C, r)`	the epicycle
S′	point	`C + polar(r, lamA)`	sun
λ_S	observable	`direction(O, S')`	true longitude

The astronomical content is in the last column. In the eccentric form the sun moves uniformly on a circle whose centre is not the earth; seen from the earth it moves unequally, fastest at the perigee. In the epicyclic form the centre of a small circle moves uniformly about the earth while the sun moves on the small circle at the same rate in the opposite sense, so that the radius to the sun keeps its direction among the stars. In both,

the mean anomaly $\bar{\kappa}$ is the angle that grows uniformly with time; the equation of centre is the difference between where the sun would be seen if it moved uniformly about the earth and where it is seen.

The diagram that shows both is a *superposition*: the two models in one plane, with one earth, one apsidal direction and one mean anomaly. Nothing else is identified. In particular the two suns, S of the eccentric form and S′ of the epicyclic, are two points. That they are one is what has to be shown, and it is not true of the superposition as such: it needs the two hypotheses the passage states in passing, r = e and R′ = R.

3.3 The argument

Figure 5 displays the argument. Its outermost claim is that the two suns are one point. It is argued by cases according to where the sun stands.

The two forms place the sun at the same point when R' = R and r = e

GIVEN

H1 the deferent has the radius of the eccentric
R′ = R

H2 the epicycle has the eccentricity as its radius
r = e

ARGUMENT

1 **the sun of the eccentric form and the sun of the epicyclic form are one point** — *cases* ✓
S = S′

- 1.1 S lies on the eccentric — *construction* ∘
  ES = R from constr. S
- 1.2 C lies on the deferent — *construction* ∘
  OC = R′ from constr. C
- 1.3 OE is the eccentricity — *construction* ∘
  OE = e from constr. E
- 1.4 S' lies on the epicycle — *construction* ∘
  CS′ = r from constr. S'
- 1.5 **Case: off the apsidal line, OESC is a parallelogram** ✓
  when OES a triangle from 1.5.7
  - 1.5.1 **ES and OC are parallel** ✓
    ES ∥ OC from 1.5.1.2
    - 1.5.1.1 both angles are the mean anomaly $\bar{\kappa}$, counted from the apsidal line — *construction* ∘
      ∠AES = ∠A′OC from constr. A, constr. S, constr. A′, constr. C
    - 1.5.1.2 the apsidal line falls on ES and OC and makes equal angles with them on the same side — *Elements I.28* ✓
      ES ∥ OC from 1.5.1.1
  - 1.5.2 ES = R = R' = OC — *common notion 1* ✓
    ES = OC from 1.1, 1.2, H1
  - 1.5.3 **OE and CS are parallel: OESC is a parallelogram** ✓
    OE ∥ CS from 1.5.3.1
    - 1.5.3.1 the lines joining the ends of the equal and parallel ES and OC — *Elements I.33* ✓
      OE ∥ CS from 1.5.1, 1.5.2
  - 1.5.4 and they are equal — *Elements I.33* ✓
    OE = CS from 1.5.1, 1.5.2
  - 1.5.5 CS = OE = e = r = CS' — *common notion 1* ✓
    CS = CS′ from 1.5.4, 1.3, H2, 1.4
  - 1.5.6 the radius of the epicycle keeps the direction of the apsidal line — *construction* ∘
    CS′ ∥ OE from constr. S′, constr. E
  - 1.5.7 S and S' lie from C along the parallel to OE, the same way, at the same distance — *lemma same ray* ✓
    S = S′ from 1.5.3, 1.5.6, 1.5.5
- 1.6 **Case: at the apogee the four points lie on the apsidal line, E between O and S** ✓
  when O, E, S on one line; E between O and S from 1.6.7
  - 1.6.1 OS = OE + ES — *lemma segment addition* ✓
  - 1.6.2 OS' = OC + CS' — *lemma segment addition* ✓
    OS′ = OC + CS′
  - 1.6.3 e + R = R' + r — *algebra* ≈
    OS = OS′ from 1.6.1, 1.6.2, 1.1, 1.2, 1.3, 1.4, H1, H2
  - 1.6.4 S lies on the apsidal line — *lemma one line* ✓
    OS ∥ OE from the case
  - 1.6.5 and so does S' — *construction* ∘
    O, E, S′ on one line from constr. C, constr. S′, constr. E
  - 1.6.6 OS′ ∥ OE — *lemma one line* ✓
    from 1.6.5
  - 1.6.7 S and S' lie from O along the apsidal line, the same way, at the same distance — *lemma same ray* ✓
    S = S′ from 1.6.4, 1.6.6, 1.6.3

1.7 **Case: at the perigee the four points lie on the apsidal line, O between E and S** ✓
when O, E, S on one line; O between E and S from 1.7.7

1.7.1 ES = EO + OS — *lemma segment addition* ✓

1.7.2 CO = CS' + S'O — *lemma segment addition* ✓
CO = CS′ + S′O

1.7.3 R − e = R' − r — *algebra* ≈
OS = OS′ from 1.7.1, 1.7.2, 1.1, 1.2, 1.3, 1.4, H1, H2

1.7.4 S lies on the apsidal line — *lemma one line* ✓
OS ∥ OE from the case

1.7.5 and so does S' — *construction* ∘
O, E, S′ on one line from constr. C, constr. S′, constr. E

1.7.6 OS′ ∥ OE — *lemma one line* ✓
from 1.7.5

1.7.7 S and S' lie from O along the apsidal line, the same way, at the same distance — *lemma same ray* ✓
S = S′ from 1.7.4, 1.7.6, 1.7.3

THEREFORE

∴ **in every state the two forms have the sun at one point** ✓
S = S′ from 1

Figure 5 The equivalence of the two forms, reconstructed (`fr:astra/solar-equivalence-equal-radii`). H1 and H2 are the hypotheses. Sub-arguments and cases are indented; a case states under "when" the positions it covers.

Off the apsidal line (case 1.5). This is the case of the printed figure and of the sentence "complete parallelogram OESC". The angles AES and A′OC are both the mean anomaly, so ES and OC are parallel (Elements I.28: "if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side … the straight lines will be parallel" [5, vol. I, p. 309]). They are equal, since the one is R and the other R′. So the lines that join their ends, OE and CS, are equal and parallel (I.33: "the straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel", p. 322): OESC is a parallelogram. The sun of the epicyclic form lies from C in the direction of the apsidal line at the distance r, and r = e = OE = CS. So S and S′ lie from C along the same parallel, the same way, at the same distance, and are one point.

At the apogee and at the perigee (cases 1.6 and 1.7). When the mean anomaly is 0° or 180°, the points O, E, S and C lie on the apsidal line. There is no parallelogram, and I.28 and I.33 have nothing to apply to (Figure 6). The argument is another one. At the apogee E lies between O and S, so OS = OE + ES = e + R; and C lies between O and S′, so OS′ = OC + CS′ = R′ + r. Under the two hypotheses these are equal, and both suns lie on the apsidal line on the same side of the earth: they are one point. At the perigee the lengths are subtracted: OS = R − e and OS′ = R′ − r.

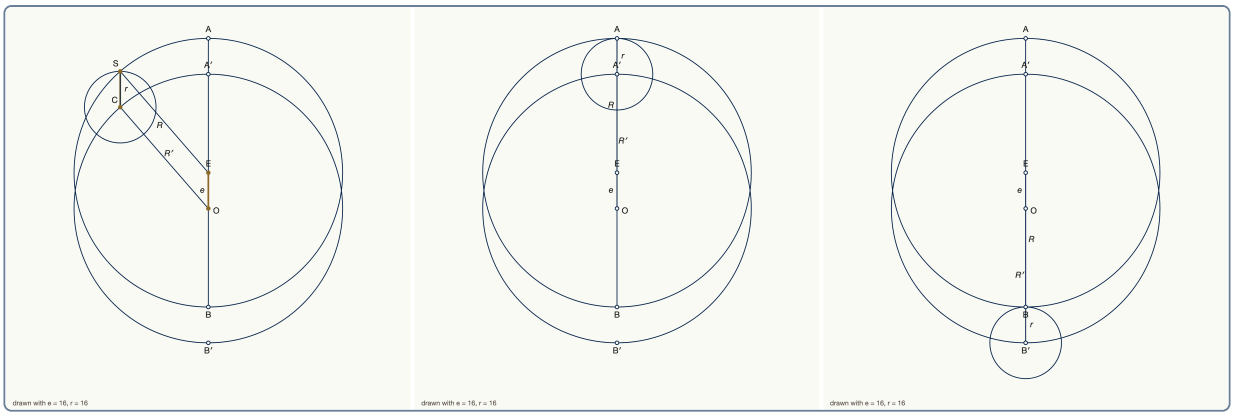


Figure 6 The three cases in the figure. Left, off the apsidal line: the parallelogram OESC with OE and CS in gold. Middle, at the apogee, and right, at the perigee: the four points on one line.

The printed figure shows the first case. The prose names the parallelogram. Neither says that the argument has two more cases. A reader supplies them without noticing; a program does not, and when the argument was first written without them it reported the two positions as states in which the cited propositions do not apply.

3.4 What the checks say

The argument passes all four checks. Of its twenty-six steps, sixteen are instances of propositions, lemmas or Common Notion 1, eight are read off the two constructions, two are calculations. Three further results are findings about the passage.

Both hypotheses are needed. With R′ = R lifted, or with r = e lifted, the conclusion fails. The passage states both, in a subordinate clause.

None is used tacitly. For every step the program determines which hypotheses it needs and whether they are cited on the way to it. They are. A version of the argument in which the step CS = CS′ did not cite r = e was reported as resting on a tacit premise.

What the constructions give without any hypothesis. Eight steps hold for every pair of models, whatever their radii: that S lies on the eccentric, that the two angles are the mean anomaly, that the radius of the epicycle keeps the direction of the apsidal line. These are what the two definitions of the models contribute. The hypotheses enter in four places only: twice in the first case, once in each of the others.

3.5 What is equivalent, and what is not

The conclusion is an equivalence *in position*: in every state the two forms place the sun at one point. It holds under the two hypotheses. Swerdlow goes on to the weaker case in which only the ratios are equal, R′/r = R/e; then the two suns lie on one ray from the earth at different distances, an equivalence *in direction*, with a different argument (similar triangles, Elements VI.6). And the equivalence is not an identity. The two models are different geometries. They agree in where the sun is and differ in how it comes to be there, which is the difference Swerdlow says "is recognized only by Kepler" [4, p. 20], [6].

4 The two cases compared

Table 4 The two reconstructions side by side.

	Euler, tide-raising force	Ptolemy's two forms, in Swerdlow
what the model contains	points and forces defined on them; one angle varies	two kinematic models superposed; one angle varies
hypotheses	none; a small ratio	two equalities of lengths
kind of conclusion	approximate, to a stated order	exact
rules	one proposition of Euclid, expansions, calculation	two propositions of Euclid, lemmas, Common Notion 1
structure	nine parts in sequence	one claim in three cases, with sub-arguments
what the check found	an editor's emendation that is wrong	two cases that figure and prose leave out

In both, the finding concerns something the text shows only in part: a formula whose derivation stands on the page, and a figure that shows one position out of all.

5 Limits

The checks test truth on samples and match rules as schemas; they are not formal proofs. In Euler's argument the two exact components are confirmed against their geometrical definition, but Euler's own way to them — the resolution along MQ and MP, the rectangle MQOP, the similar triangles QVM and MPC — is not followed step by step; it is named in the gloss of the two steps. The expansion is tested numerically, not derived term by term. In the equivalence, the two cases on the apsidal line use two lemmas of the author; what they rest

on in Euclid is said in the rule table. Swerdlow's page is the witness for Ptolemy's model; the *Almagest* itself is not among the texts available to us in addressable form.

Availability

The texts are in the Famulus workspace `famulus-astronomica`: `examples/euler/` (model, derivation, the tested emendation, figure) and `examples/solar/` (the two models, their superposition, the derivation, figures); the reading file `readings/euler-e57-tide.yaml`. The interactive form of both arguments is the element `<astra-argument>`; the HTML version of this paper contains it in place of Figure 2 and Figure 5. Euler is cited by the pages of the Famulus collection `enestroem`, document `E57` (page n of the collection is page n + 19 of Opera omnia II 31); Swerdlow by the collection `kepler-sec`.

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